72

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CONTENTS TO VOL. XXXV.

MESSENGER OF MATHEMATICS.

SOME CONTRIBUTIONS TO THE THEORY OF TWO ELECTRIFIED SPHERES.

By *T. J. I'a. Bromwich*, Queen's College, Galway.

ALTHOUGH the problem of the electric distribution on two conducting spheres has been handled very often, yet the discussion of the problem when the two spheres move up to touch one another involves certain difficulties which do not seem to have been adequately treated. For example, the force between two equal spheres in contact has been calculated by Lord Kelvin,* in terms of a double series; but unfortunately this series does not converge absolutely, and consequently the value of the series depends upon the mode of summation.

In fact, as I have proved elsewhere,† the value obtained by Lord Kelvin *cannot* be interpreted as the force between the sets of images in the two spheres: the latter force is indeterminate and oscillates between two limits which are approximately $(.1364)V^2$ and $(.0114)V^2$ if V is the potential of the two spheres. Lord Kelvin's value is the mean between these two limits, or is approximately $(.0739)V^2$.

That the mode of summation adopted by Lord Kelvin does give the correct value for the force between the two spheres may be proved by considerations into which I shall not enter here: but I shall give a different investigation (§ 1) leading to the same value.

Another problem in this connection is the calculation of the surface-density just before contact, at the points which afterwards touch. The correct value for this was given first by Kirchhoff, after having been calculated incorrectly by

* *Phil Mag.*, April and August, 1853; *Reprint of Electrical Papers*, No. VI.
† *Proc. Lond. Math. Soc.*, (2), Vol. 2, 1904, p. 175.

earlier writers. I give in § 2 an alternative investigation, suggested by the work in § 1.

§ 1. *Force between two equal spheres in contact.*

From the investigations given by the authors quoted above,* there is no difficulty in seeing that for two equal spheres of radius a, whose centres are at a distance b apart, the coefficient of capacity is

$$a \sinh\theta \left[\frac{1}{\sinh\theta} + \frac{1}{\sinh 3\theta} + \frac{1}{\sinh 5\theta} + \dots\right].$$

and that the coefficient of induction is

$$-a \sinh\theta \left[\frac{1}{\sinh 2\theta} + \frac{1}{\sinh 4\theta} + \frac{1}{\sinh 6\theta} + \dots\right],$$

where θ is given by the equation

$$b = 2a \cosh\theta.$$

It follows that, when both the spheres are at potential V, the potential energy of the system is

$$W = aV^2 \sinh\theta \left[\frac{1}{\sinh\theta} - \frac{1}{\sinh 2\theta} + \frac{1}{\sinh 3\theta} - \frac{1}{\sinh 4\theta} + \dots\right].$$

the rearrangement of terms being permissible in consequence of the absolute convergence of the series.

Since the potential energy is expressed in terms of a, b, V, it follows that the force tending to *increase* b is $+\dfrac{dW}{db}$; of course if W were expressed in terms of a, b, and the *charge*, this force would be $-\dfrac{dW}{db}$.

We have now to calculate the value of this differential coefficient for $b = 2a$, or for $\theta = 0$. By definition this desired value will be

$$X = \lim_{b=2a} \frac{W - W_0}{b - 2a} = \lim_{\theta=0} \frac{W - W_0}{a\theta^2},$$

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* Reference may also be made to Maxwell's *Electricity and Magnetism* Vol. I., Chap. 11, or to Kirchhoff's *Vorlesungen*, Bd. 3, pp. 61–89.

where*

$$W_0 = \lim_{\theta=0} W.$$

To evaluate the force X, we must first examine the behaviour of the function

$$f(\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sinh n\theta}$$

for small values of θ.

Since

$$\frac{1}{\sinh n\theta} = \frac{1}{n\theta}\int_0^\infty \frac{\cos(2n\theta x/\pi)}{\cosh^2 x}\,dx$$

(see § 3 below), it is clear that

$$f(\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\theta}\int_0^\infty \frac{\cos(2n\theta x/\pi)}{\cosh^2 x}\,dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\theta}\left(\int_0^\lambda + \int_\lambda^\infty\right)\frac{\cos(2n\theta x/\pi)}{\cosh^2 x}\,dx,$$

where $\lambda = 1/\theta^{\frac{1}{2}}$.

Now

$$\int_\lambda^\infty \frac{\cos(2n\theta x/\pi)}{\cosh^2 x}\,dx = \frac{1}{\cosh^2\lambda}\int_\lambda^\mu \cos(2n\theta x/\pi)\,dx, \qquad (\mu > \lambda),$$

by the second theorem of the mean; and hence

$$\left|\int_\lambda^\infty \frac{\cos(2n\theta x/\pi)}{\cosh^2 x}\,dx\right| < 4\frac{\pi}{n}\lambda^2 e^{-2\lambda}.$$

It therefore follows that

$$f(\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\theta}\int_0^\lambda \frac{\cos(2n\theta x/\pi)}{\cosh^2 x}\,dx + g(\theta),$$

where

$$|\,g(\theta)\,| < 4\pi\lambda^2 e^{-2\lambda}\sum_{n=1}^{\infty}(1/n^2\theta),$$

or

$$|\,g(\theta)\,| < \tfrac{2}{3}\pi^3\lambda^4 e^{-2\lambda}.$$

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* It is assumed that W is continuous up to and including the value $\theta = 0$; this can be proved by physical considerations.

Again from an investigation given by Mr. G. H. Hardy,* it appears that

$$\int_0^{\lambda} \log (4 \cos^2 \tfrac{1}{2} mx)\, \phi (x)\, dx = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^{\lambda} \phi (x) \cos nmx dx,$$

provided that $\phi(x)$ steadily decreases as x increases, and that the series on the right is convergent. In the case which we are here concerned with, $\phi(x) = 1/\cosh^2 x$; and the convergence of the series follows from the fact that $f(\theta)$, $g(\theta)$ both converge absolutely.

Hence

$$f(\theta) = \frac{1}{2\theta} \int_0^{\lambda} \frac{\log (4 \cos^2 \theta x/\pi)}{\cosh^2 x}\, dx + g(\theta).$$

In the last integral $\theta x/\pi$ ranges from 0 to $\theta^{\frac{1}{2}}/\pi$; and consequently, since θ is regarded as small, the angle $\theta x/\pi$ may also be taken as small.

Now, when y is sufficiently small, $\tan y$ lies between y and $y + y^3$; for we have

$$\frac{d}{dy} (\tan y - y) = \tan^2 y;$$

also, since $(\tan y/y)$ increases with y, we see that

$$\tan y/y < 4/\pi < \sqrt{3}, \quad \text{if } 0 < y < \tfrac{1}{4}\pi;$$

thus

$$0 < \frac{d}{dy} (\tan y - y) < 3y^2.$$

Consequently, since $(\tan y - y)$ is zero for $y = 0$, its value lies between 0 and y^3, provided that $0 < y < \frac{1}{4}\pi$. But

$$\frac{d}{dy} (\log \sec^2 y) = 2 \tan y,$$

so that $\quad 2y < \dfrac{d}{dy} (\log \sec^2 y) < 2y + 2y^3, \quad \text{if } 0 < y < \tfrac{1}{4}\pi,$

and therefore

$$y^2 < \log \sec^2 y < y^2 + \tfrac{1}{2} y^4.$$

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* *Quarterly Journal of Mathematics*, Vol. XXXIV., 1902, p. 28. I am indebted to Mr. Hardy for the suggestion that this result might prove useful in the examination of $f(\theta)$.

Now, in the integral $\theta x/\pi$ ranges from 0 to $\theta^{\frac{1}{2}}/\pi$ (which may be supposed less than $\frac{1}{4}\pi$, since θ is *small*), and consequently we may use the inequalities

$$\log 4 - (\theta x/\pi)^2 > \log(4\cos^2\theta x/\pi) > \log 4 - (\theta x/\pi)^2 - \tfrac{1}{2}(\theta x/\pi)^4.$$

Hence

$$\int_0^\lambda \frac{\log 4 - (\theta x/\pi)^2}{\cosh^2 x}\,dx - \int_0^\lambda \frac{\log(4\cos^2\theta x/\pi)}{\cosh^2 x}\,dx < \tfrac{1}{2}\int_0^\lambda \frac{(\theta x/\pi)^4}{\cosh^2 x}\,dx.$$

Now

$$\int_0^\lambda \frac{dx}{\cosh^2 x} = \tanh\lambda = 1 - \frac{2}{1+e^{2\lambda}},$$

and, by § 3,

$$\int_0^\lambda \frac{x^2 dx}{\cosh^2 x} = \int_0^\infty \frac{x^2 dx}{\cosh^2 x} - \int_\lambda^\infty \frac{x^2 dx}{\cosh^2 x} = \frac{\pi^2}{12} - \int_\lambda^\infty \frac{x^2 dx}{\cosh^2 x}.$$

Further

$$\int_\lambda^\infty \frac{x^2 dx}{\cosh^2 x} < \int_\lambda^\infty 4x^2 e^{-2x}\,dx < 5\lambda^2 e^{-2\lambda},*$$

and

$$\int_0^\lambda \frac{x^4 dx}{\cosh^2 x} < \int_0^\lambda 4x^4 e^{-2x} dx < 3.$$

Combining these results, it appears that

$$\int_0^\lambda \frac{\log(4\cos^2\theta x/\pi)}{\cosh^2 x}\,dx$$

differs from $(\log 4 - \frac{1}{12}\theta^2)$ by less than

$$(2\log 4 + 5\theta/\pi^2)e^{-2\lambda} + \tfrac{3}{2}(\theta/\pi)^4.$$

Consequently

$$\theta f(\theta) = \tfrac{1}{2}\int_0^\lambda \frac{\log(4\cos^2\theta x/\pi)}{\cosh^2 x}\,dx + \theta g(\theta)$$
$$= \log 2 - \tfrac{1}{24}\theta^2 + h(\theta),$$

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* For we have $\displaystyle\int_\lambda^\infty 4x^2 e^{-2x}dx = (2\lambda^2 + 2\lambda + 1)e^{-2\lambda}$,

which is less than $5\lambda^2 e^{-2\lambda}$, if $\lambda > 1$ as we assume.

Also $\displaystyle\int_0^\infty 4x^4 e^{-2x} dx = 3.$

where

$$|\,h(\theta)\,| < (\log 4 + \tfrac{5}{2}\theta/\pi^2)e^{-2\lambda} + \tfrac{3}{4}(\theta/\pi)^4 + \tfrac{2}{3}\pi^3\lambda^4 e^{-2\lambda}.$$

Now $W = aV^2 \sinh\theta f(\theta)$ so that

$$W_0 = \lim_{\theta=0} W = aV^2 \lim_{\theta=0} \theta f(\theta) = aV^2 \log 2.$$

Thus

$$\lim_{\theta=0} \frac{W - W_0}{a\theta^2} = V^2 \lim_{\theta=0} \left[\log 2\left(\frac{\sinh\theta}{\theta^3} - \frac{1}{\theta^2}\right) - \frac{1}{24}\frac{\sinh\theta}{\theta}\right]$$

$$+ V^2 \lim_{\theta=0} \frac{h(\theta)}{\theta^4}\frac{\sinh\theta}{\theta}.$$

But

$$\lim_{\theta=0}\left(\frac{\sinh\theta}{\theta^3} - \frac{1}{\theta^2}\right) = \frac{1}{6}, \quad \lim_{\theta=0}\frac{\sinh\theta}{\theta} = 1;$$

and

$$|\,h(\theta)/\theta^2\,| < (\lambda^4 \log 4 + \tfrac{5}{2}\lambda^2/\pi^2)\,e^{-2\lambda} + \tfrac{3}{4}\theta^3/\pi^4 + \tfrac{2}{3}\pi^3\lambda^6 e^{-2\lambda},$$

so that $\lim\limits_{\theta=0} h(\theta)/\theta^2 = 0$; because λ tends to infinity as θ tends to zero.

Hence the force to be found is

$$X = \lim_{\theta=0}\frac{W - W_0}{a\theta^2} = V^2\left(\tfrac{1}{6}\log 2 - \tfrac{1}{24}\right),$$

as given by Lord Kelvin.*

§ 2 *Surface-density at the nearest points of the two spheres.*

Write $c^2 = \frac{1}{4}b^2 - a^2$, then the potential at any point of the line of centres between the two spheres can be expressed in the form

$$U = V + V\cosh\phi\left[\sum_{n=-\infty}^{+\infty}\frac{1}{\cosh\{\phi + (2n-1)\theta\}} - \sum_{n=-\infty}^{+\infty}\frac{1}{\cosh(\phi + 2n\theta)}\right],$$

* The approximate value of $\frac{1}{6}(\log 2 - \frac{1}{4})$ is .0739 to four decimal places.

where $c \tanh \phi$ is the distance of the point in question from the mid-point of the line of centres; the value of ϕ is $\pm\frac{1}{2}\theta$ at the two surfaces.

To find the surface density at the points indicated, let us write

$$F(\phi) = \sum_{n=-\infty}^{+\infty} \operatorname{sech}(\phi + 2n\theta),$$

then

$$U = V + V\cosh\phi\,[F(\phi - \theta) - F(\phi)],$$

so that

$$\sigma = \frac{\cosh^2\frac{1}{2}\theta}{4\pi c}\left(\frac{\partial U}{\partial \phi}\right)_{\phi=\frac{1}{2}\theta} = -\frac{V\cosh^3\frac{1}{2}\theta}{2\pi c}\,F'(\tfrac{1}{2}\theta).$$

We proceed now to the discussion of the function $F(\phi)$, which is, incidentally, an elliptic function with periods 2θ and $2\pi i$. In fact by the use of certain theta-function formulæ we could dispense with the following investigation; but as an independent method, it has some points of interest.

Now, by § 3, we have

$$\frac{2}{\pi}\int_0^\infty \frac{\cos[2(\phi + 2n\theta)x/\pi]}{\cosh x}\,dx = \frac{1}{\cosh(\phi + 2n\theta)},$$

so that

$$F(\phi) = \lim_{p=\infty} \frac{2}{\pi}\int_0^\infty \sum_{n=-p}^{+p} \frac{\cos[2(\phi + 2n\theta)x/\pi]}{\cosh x}\,dx$$

$$= \lim_{p=\infty} \frac{2}{\pi}\int_0^\infty \frac{\sin[2(2p+1)\theta x/\pi]\cos(2\phi x/\pi)}{\sin(2\theta x/\pi)\cosh x}\,dx$$

$$= \lim_{p=\infty} \frac{1}{\theta}\int_0 \frac{\sin(2p+1)x}{\sin x} \cdot \frac{\cos(\phi x/\theta)}{\cosh(\pi x/2\theta)}\,dx.$$

Let us denote the function

$$\frac{\cos(\phi x/\theta)}{\cosh(\pi x/2\theta)}$$

by $G(x)$; and it is to be observed that

$$|\,G(x)\,| < 2e^{-\pi x/2\theta}.$$

Then, since the integral

$$\int_c^\infty \frac{\sin(2p+1)x}{\sin x}\,G(x)\,dx$$

is convergent, its value is the same as that of the limit

$$\lim_{q=\infty}\left[\int_0^{\frac{1}{2}\pi}+\int_{\frac{1}{2}\pi}^{\pi}+\int_{\pi}^{\frac{3}{2}\pi}+\ldots+\int_{q\pi}^{(q+\frac{1}{2})\pi}\right].$$

Now

$$\int_{\frac{1}{2}\pi}^{\pi}\frac{\sin(2p+1)x}{\sin x}\,G(x)\,dx=\int_0^{\frac{1}{2}\pi}\frac{\sin(2p+1)x}{\sin x}\,G(\pi-x)\,dx,$$

$$\int_{\pi}^{\frac{3}{2}\pi}\frac{\sin(2p+1)x}{\sin x}\,G(x)\,dx=\int_0^{\frac{1}{2}\pi}\frac{\sin(2p+1)x}{\sin x}\,G(\pi+x)\,dx,$$

and so on; thus we find

$$\int_0^{\infty}\frac{\sin(2p+1)x}{\sin x}\,G(x)\,dx$$

$$=\lim_{q=\infty}\int_0^{\frac{1}{2}\pi}\frac{\sin(2p+1)x}{\sin x}\left[G(x)+G(\pi-x)+G(\pi+x)\right.$$
$$\left.+\ldots+G(q\pi+x)\right]dx.$$

But the series (to infinity)

$$H(x)=G(x)+G(\pi-x)+G(\pi+x)+G(2\pi-x)+G(2\pi+x)+\ldots$$

has its terms less, in absolute value, than those of the series

$$2+2e^{-(\pi^2/4\theta)}+2e^{-2(\pi^2/4\theta)}+2e^{-3(\pi^2/4\theta)}+\ldots,$$

when x ranges from 0 to $\frac{1}{2}\pi$, inclusive.

Thus the series $H(x)$ converges uniformly when x varies between these limits; and consequently

$$\lim_{q=\infty}\int_0^{\frac{1}{2}\pi}\frac{\sin(2p+1)x}{\sin x}\left[G(x)+\ldots+G(q\pi+x)\right]dx$$

$$=\int_0^{\frac{1}{2}\pi}\frac{\sin(2p+1)x}{\sin x}\,H(x)\,dx;$$

or

$$\int_0^{\infty}\frac{\sin(2p+1)x}{\sin x}\,G(x)\,dx=\int_0^{\frac{1}{2}\pi}\frac{\sin(2p+1)x}{\sin x}\,H(x)\,dx.$$

Hence

$$F(\phi)=\lim_{p=\infty}\frac{1}{\theta}\int_0^{\frac{1}{2}\pi}\frac{\sin(2p+1)x}{\sin x}\,H(x)\,dx.$$

Since $H(x)$ is uniformly convergent within the limits of integration, $H(x)$ must represent a continuous function of x; further, if the series for $H(x)$ be differentiated term by term, the series of derivates is uniformly convergent, as is seen by comparison with the series

$$[(2\phi+\pi)/\theta)]\left[1+e^{-(\pi^2/4\theta)}+e^{-2(\pi^2/4\theta)}+e^{-3(\pi^2/4\theta)}+\ldots\right].$$

Thus $H(x)$ has a continuous derivate within the limits of integration; and consequently the theory of Dirichlet's integral shews that

$$\lim_{p=\infty}\int_0^{\frac{1}{2}\pi}\frac{\sin(2p+1)x}{\sin x}H(x)\,dx=\tfrac{1}{2}\pi H(0).$$

Hence

$$F(\phi)=\frac{\pi}{2\theta}\left[G(0)+2G(\pi)+2G(2\pi)+\ldots\right]$$

$$=\frac{\pi}{2\theta}\left[1+2\frac{\cos(\pi\phi/\theta)}{\cosh(\pi^2/2\theta)}+2\frac{\cos(2\pi\phi/\theta)}{\cosh(2\pi^2/2\theta)}+\ldots\right].$$

This series for $F(\phi)$ converges very rapidly when θ is small; and the series of derivates with respect to ϕ is easily seen to be uniformly convergent for all values of ϕ.

Thus

$$F'(\tfrac{1}{2}\theta)=-\left(\frac{\pi}{\theta}\right)^2\left[\frac{1}{\cosh(\pi^2/2\theta)}-\frac{1}{\cosh(3\pi^2/2\theta)}+\ldots\right],$$

and when θ is small, the sum is given with considerable accuracy by the approximation

$$-2(\pi/\theta)^2e^{-\pi^2/2\theta}.$$

The corresponding approximation for the surface-density is

$$\sigma=(\pi V/c\theta^2)\,e^{-\pi^2/2\theta}.$$

But $c=a\sinh\theta$, so that we may replace this by

$$\sigma=(\pi V/a\theta^3)\,e^{-\pi^2/2\theta},$$

where, approximately, $\theta^2=(b-2a)/a$.

The above is Kirchhoff's result.*

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* *Ges. Abhandlungen*, p. 90; in Kirchhoff's notation $\delta=-2\theta$, and $a=1$.

§ 3. *Calculation of the definite integrals used above.*

Since

$$\frac{1}{e^x+1} = e^{-x} - e^{-2x} + e^{-3x} - \ldots + e^{-(2n-1)x} - \frac{e^{-2nx}}{1+e^x},$$

and

$$\int_0^\infty e^{-px} \sin mx\, dx = \frac{m}{p^2+m^2},$$

it follows that

$$\int_0^\infty \frac{\sin mx}{e^x+1}\, dx = \frac{m}{1+m^2} - \frac{m}{2^2+m^2} + \frac{m}{3^2+m^2}$$

$$-\ldots+ \frac{m}{(2n-1)^2+m^2} - R_n,$$

where

$$R_n = \int_0^\infty \frac{e^{-2nx}}{1+e^x} \sin mx\, dx.$$

Now it is plain that in numerical value

$$|R_n| < \int_0^\infty e^{-2nx}\, dx,$$

or

$$|R_n| < 1/2n.$$

Hence $\lim\limits_{n=\infty} R_n = 0$, and accordingly

$$\int_0^\infty \frac{\sin mx}{e^x+1}\, dx = \sum_{p=1}^{p=\infty} \frac{(-1)^{p-1} m}{p^2+m^2};$$

an equation which can be established also by considerations of uniform convergence, although the foregoing method seems to be easier.

Now if $|m|$ is less than an arbitrary fixed positive number m_0, the series $\Sigma(-1)^{p-1} m/(p^2+m^2)$ has the property that its terms are less in absolute value than those of the series $\Sigma m_0/p^2$; hence by Weierstrass's well-known test*, the series $\Sigma(-1)^{p-1} m/(p^2+m^2)$ converges absolutely and uniformly for all values of m such that $|m| \leq m_0$.

In consequence of the absolute convergence we may

* *Ges. Werke*, Bd. 2, p. 202.

transform the last equation to

$$\int_0^\infty \frac{\sin mx}{e^x+1}\,dx = \sum_{p=1}^{\infty} \frac{m}{p^2+m^2} - 2\sum_{p=1}^{\infty} \frac{m}{4p^2+m^2},$$

and this transformation leaves* the series still uniformly convergent for the same values of m, Weierstrass's test being still applicable.

The series being uniformly convergent, we see that

$$\sum_{p=1}^{\infty} \frac{m}{p^2+m^2} = \tfrac{1}{2}\frac{\partial}{\partial m}\log \prod_{p=1}^{\infty}\left(1+\frac{m^2}{p^2}\right),$$

and

$$\sum_{p=1}^{\infty} \frac{m}{4p^2+m^2} = \tfrac{1}{2}\frac{\partial}{\partial m}\log \prod_{p=1}^{\infty}\left(1+\frac{m^2}{4p^2}\right).$$

Also

$$\prod_{p=1}^{\infty}\left(1+\frac{m^2}{p^2}\right) = \frac{\sinh(m\pi)}{m\pi},$$

so that

$$\sum_{p=1}^{\infty} \frac{m}{p^2+m^2} = \tfrac{1}{2}\pi\coth(m\pi) - \frac{1}{2m},$$

and consequently

$$\sum_{p=1}^{\infty} \frac{m}{p^2+m^2} - \sum_{p=1}^{\infty} \frac{2m}{4p^2+m^2} = \tfrac{1}{2}\pi\coth(m\pi) - \frac{1}{2m}$$

$$- \left[\tfrac{1}{2}\pi\coth\left(\tfrac{1}{2}m\pi\right) - \frac{1}{m}\right] = \tfrac{1}{2}\left[\frac{1}{m} - \frac{\pi}{\sinh(m\pi)}\right].$$

Hence

$$\int_0^\infty \frac{\sin mx}{e^x+1}\,dx = \tfrac{1}{2}\left[\frac{1}{m} - \frac{\pi}{\sinh(m\pi)}\right].$$

But, integrating by parts, the integral is also equal to

$$\frac{1}{2m} - \frac{1}{m}\int_0 \frac{e^x\cos(mx)\,dx}{(e^x+1)^2},$$

so that

$$\frac{\pi}{\sinh(m\pi)} = \frac{2}{m}\int_0^\infty \frac{e^x\cos(mx)}{(e^x+1)^2}\,dx = \frac{2}{m}\int_0^\infty \frac{\cos(mx)}{4\cosh^2\left(\frac{1}{2}x\right)}\,dx.$$

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* That such a transformation might affect the uniform convergence appears from an example constructed by Prof. M. Bôcher, *Annals of Mathematics*, (2), Vol. 4, 1903, p. 159. But, whenever Weierstrass's test can be applied, the series remains uniformly convergent in any arrangement.

By writing $n\theta$ for $m\pi$ and $2y$ for x, this becomes

$$\frac{1}{\sinh(n\theta)} = \frac{1}{n\theta}\int_0^\infty \frac{\cos(2n\theta y/\pi)}{\cosh^2 y}\,dy,$$

which is the first result used in § 1.

By a similar argument we prove that

$$\int_0^\infty \frac{x^2dx}{\cosh^2 x} = \int_0^\infty \frac{xdx}{e^x+1} = \sum_{p=1}^\infty \frac{1}{p^2} - 2\sum_{p=1}^\infty \frac{1}{4p^2} = \frac{\pi^2}{12},$$

which is the second result.

An alternative method for finding the integral $\int_0^\infty \frac{\sin(mx)}{e^x+1}\,dx$ is to consider the complex integral $\int \frac{e^{imz}}{e^z+1}\,dz$, taken round a rectangle in the z-plane, the rectangle being bounded by the positive halves of the real axis and of the line $y=2\pi$, by the intercept on the imaginary axis and a parallel line at infinity.* The point $z=\pi i$ is to be excluded by a small semicircle in the usual way.

By a similar process we can calculate the integral

$$\int_0^\infty \frac{\cos mx}{\cosh x}\,dx;$$

but this is most easily obtained by taking the complex integral $\int \frac{e^{imz}}{\cosh z}\,dz$ round the rectangle made up of the real axis, the line $y=\pi$, and two lines at infinity. In this way we get

$$2(1+e^{-m\pi})\int_0^\infty \frac{\cos mx}{\cosh x}\,dx = 2\pi e^{-\frac{1}{2}m\pi},$$

the right-hand side being found by contracting the rectangle to a small circle round the point $z=\frac{1}{2}i\pi$, which is the only singularity of the integrand within the rectangle.

Hence

$$\int_0^\infty \frac{\cos mx}{\cosh x}\,dx = \frac{1}{2}\frac{\pi}{\cosh(\frac{1}{2}m\pi)},$$

which is the value used in § 2.

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* This phrase is not strictly accurate, but its meaning seems to be clear enough

POWER-TABLES.—ERRATA.

By *Lt.-Col. Allan Cunningham, R.E*, Fellow of King's College, London.

[The author's acknowledgments are due to Mr. H. J. Woodall for help in preparing this Paper and in correcting the proof-sheets.]

1. *Power-Tables—Errata.* DURING some heavy factorisations of numbers of form $N=(a^n \mp 1)$, it became evident that some of the printed Power-Tables contained a good many Errata. Errors in such Tables are not easily detected, and are liable to lead to consequential errors in their use. After much trouble arising from this cause, the writer decided to undertake an examination of the principal Tables of powers *higher than the cube.* The Tables now reported on are in general those of powers *of integer numbers* only: but a few Tables of powers of (decimal) fractions are included (see Nos. 4, 5, 7, 12 below) where the number of decimals printed is sufficient to give the *accurate value* of the power, so that the Table can be used as an ordinary Table of powers of integers, by simply omitting the decimal point and all the unnecessary ciphers.

2. *Power-Tables examined.* The British Association Committee Report on Mathematical Tables, drawn up by Mr. (now Dr.) J. W. L. Glaisher, contains—(see *Brit. Assoc. Report* for 1873, p. 29)—a List of 12 Power-Tables published at various dates from 1650 to 1873, with the full Titles and a brief description of each (*op. cit.*, pp. 83 to 164).

The following is an Abstract of the Tables examined,* showing the author's name, and date† of publication, a reference to the pages on which the Power-Tables are printed, a brief indication of the nature and scope of each Table, and of the extent of the verification effected, and lastly a note of the number of Errata found. For full Titles of the Works containing the Tables, and for a description of the Tables themselves, and for detailed List of the Errata found, see Appendix I.

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* These include all those in the British Association List, and also several others not in that List.

† When two dates are given, the first shows the date of the first Edition: the second date, enclosed in square brackets, as [1829], shows the date of the Edition or impression examined.

Abstract of Power-Tables examined.

No.	*Author*	*Date*	*Page*	*a*	*n*	*Verifn.*	*Err.*
1	Moore, Sir J.	1650 1660	113 *et seq* Appx. 1 to 48	1 to 200 200 to 300	1 to 6 1 to 4		
2	Dodson, Jas.	1747	42 44	2 1 to 9	0 to 29 1 to 20	all all	0 0
3	Hill, J.	1764	159 to 161 156	2 2	0 to 143 144	all all	0 0
4	Lambert J.	1770	118 to 120 202 to 206	2, 3, 5 0.01 to 1.00	1 to 70, 50, 50 1 to 4	all all	0 2
5	Schulze, J. C.	1778	278 to 281	0.01 to 1.00	1 to 4	all	2
6	Hutton, Ch.	1781	101	1 to 99	1 to 10	all	3
7	Callet, Fr.	1795 [1829]	13	0.5	1 to 60	all	0
8	Vega, Geo.	1797		2, 3, 5 1 to 100	1 to 45, 36, 27 1 to 9		
9	Barlow, P.	1814	170 to 173 176 to 184	1 to 100 100 to 999	1 to 10 4 & 5	all all	4 41
10	Kulik, J. P.	1825	286	2 3 & 5	10 to 71 7 to 37	all all	1 0
11	Jahn, G. A.	1839	241	1 to 50	4 to 9	all	25
12	Vega, Geo. [Ed. Hülsse]	1840 [1849]	466 578 to 581 582 to 585	2, 3, 5 0.01 to 1.00 1 to 100	1 to 45, 36, 27 1 to 4 1 to 9	all all all	0 2 0
13	Köhler, H. G.	1848 [1864]	355 360 to 363	2, 3, 5 1 to 100	1 to 45, 36, 27 1 to 9	all all	0 0
14	Shanks, Wm.	1853	90 to 95	2	$n = 12\nu + 1$ 13 to 721	13 to 145 253, 517, 721	0 0 0
15	Beardmore, N.	1862	86, 87	1 to 100	1, 2, 3, 5	all	1
16	Molesworth, G. L.	1862 [1882]	709	1 to 100	4 & 5	all	2
17	Rankine, W. J. M.	1866 [1867]	32	10 to 99	2 & 5	all	1
18	Hoüel, J.	1866	64	2 3, 5, 6, 7, 9 11 to 15	1 to 45 1 to 10 1 to 10	all all all	0 0 1
19	Martin, Art.	1883	32	2	$n = 2^\nu$ $\nu = 1$ to 9	all	0
20	Adams, J. C.	1890	Appx. p. xiii	2	n odd 1 to 125	all	0
21	Laffaille, J.	1896	62	1 to 9	1 to 11	all	2
22	Putnam, K. S.	1898	27	2	721 & 1024	721	0
23	Woodall H. J.	1898	27	2	103	all	0
24	Cunningham, A.	1900	viii	2	1 to 45	all	0

Abstract of (unpublished) Power-Tables examined.

No.	*Author*	*Date*	*Page*	*a*	*n*	*Verifn.*	*Err.*
25	Glaisher, J. W. L. [Unpub.]	1873?	1 to 58	0 to 999	1 to 12	part	0
26	Cunningham, A.	1875 MS.		2, 3, 4, 5	1 to 32	all	0
				6, 7, 8, 9	1 to 20	all	0
				11 to 25	1 to 16	all	0
				26, 27	1 to 10	all	0
27	Woodall, H. J.	1898 MS.		2	1 to 144	all	0
		1901 MS.		3 & 5	1 to 40	all	0

The older Tables, printed before 1800, are now so difficult to procure, as to be of little general use, although some of them contain matter not to be found in the later Tables. The following is a brief epitome of the points in which some of the Tables examined are *unique*, either in range of the base-numbers (a) or in range of powers (n).

No.	*Author.*	*Date.*	*a.*	*n.*
1.	Moore,	1650;	1 to 200,	1 to 6
2.	Dodson,	1747;	4 to 9	, 1 to 20
3.	Hill,	1767;	2	, 1 to 144
4.	Lambert,	1770;	3, 5	, 1 to 50
7.	Callet,	1795;	0.05	, 1 to 60
9.	Barlow,	1874;	1 to 999,	4 & 5
14.	Shanks,	1853;	2	, 13 to 721
25.	Glaisher, [unpub.]	1873?;	0 to 999,	1 to 12

3. *Tables of Squares and Cubes.* Tables of this sort are in general here reported on only when *included** in Tables of higher powers (shown by $n = 2, 3$ in the Abstract of Tables reported on). In Appendix II., however, a List is given of such Errata as have come under the author's notice in two such Tables (Nos. 9, 11 of the Abstract).

Of course Tables of Squares and Cubes can—if sufficiently extensive—be used for taking out higher powers of form $n = 2\nu, 3\nu$ respectively. Thus the ordinary Tables extending to 10000^2 and 10000^3, and the exceptional Tables extending to 100000^2 and 100000^3 can be used for taking out a^n to following high limits of $n = 2\nu$, or 3ν.

Table.	$a =$	2,	3,	4,	5,	6,	7 to 9,	11 to 21,	22 to 100;
Squares to 10000^2	$n = 2\nu =$	26,	16,	12,	10,	10,	8,	6,	4;
Cubes to 10000^3	$n = 3\nu =$	39,	24,	18,	15,	15,	12,	9,	6;

	$a =$	2,	3,	4,	5,	6,	7 to 9,	11 to 17,	18 to 46,	47 to 316;
Squares to 100000^2,	$n = 2\nu =$	32,	20,	16,	14,	12,	10,	8,	6,	4;
Cubes to 100000^3,	$n = 3\nu =$	48,	30,	24,	21,	18,	15,	12,	9,	6.

4. *Mode of collation.* The Tables were read together† in pairs, and a List was made of all discrepancies; these

* Tables of Squares and Cubes are so numerous and so bulky that any thorough examination of them would be a very serious matter.

† By the author's assistant (Mr. R. F. Woodward).

discrepancies were then examined by two* computers, (working independently), and the Erratum was thus allotted to the Table in fault. This mode of collation thus affords a *complete comparison* (not really a verification) of *all* the Tables to the *extent to which any pair cover the same ground.* The agreement of a pair of Tables, however, by no means ensures their accuracy: in several cases the *same error*† was found to affect two or more Tables.

5. *Appendix I.* This Appendix is divided into 27 paragraphs (numbered 1 to 27), one for each set of Tables (Nos. 1 to 27) examined. Each paragraph contains (1) the full Title of the Work containing the Tables under review; (2) a reference to the pages on which the Tables in question are contained; (3) a List of the Errata found, or of the Corrigenda required.

6. *Errata, Corrigenda.* Many of the numbers in these Tables are *very large.* In order to save space only so many figures of each number are printed as will suffice to identify the Erratum and the consequent Corrigendum: the *incorrect* and the *corrected* figures (to be substituted) are, for the sake of clearness, printed in *black figures*; the extra (correct) figures, required merely to locate the Erratum, being printed in ordinary type.

Appendix I. Power-Tables.—Errata.

1. Moore, Sir Jonas. Two works are here reported on; viz.

(1) *Moore's Arithmetic in two Bookes*, London, 1650.

The Table, Book ii., pp. 115 to 141, styled a *Canon of the Squares, Cubes, &c.* gives a^n as follows:—

a^2 & a^3 from $a = 1$ to 1000; a^4 from $a = 1$ to 300; a^5 & a^6 from 1 to 200.

The headings of the columns are curious; q, c being used to denote Square, Cube; thus

Headline.	*L* ,	*Aq* ,	*Ac* ,	*Aqq* ,	*Aqc* ,	*Acc*
Meaning.	Argument,	Square,	Cube,	4th power,	5th power,	6th power.

The pagination is very faulty;

thus, where the sequence requires 122, 123; 126, 127; 129 onward to 147,
the actual pagination is 112, 113; 116, 117; 123 onward to 141.

The column-headings (*L*, *Aq*, *Ac*, *&c.*) have been omitted on pages 120, 121, 124, 125, 128.

The typography is only fairly clear, but after $a = 300$, *i.e.* from page 132, the type is very crowded, there being 35 lines in the page, (instead of 25 as previously).

(2) *Moor's Arithmetic in two Books*, London, 1660. This appears to be a later Edition of the last, but is of larger size. The *Canon of Squares*,

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* By the author himself, and by an Assistant.

† I.e. The same errors occur in 0.36^3 and 0.71^4 in Lambert's, Schulze's, and (Hulsse's Ed. of) Vega's Tables, see Nos. 4, 5, 12 in Appendix I.

Cubes, &c., at the end if the book is similar to, and of same extent as, that in the Edition of 1650, but has *separate* pagination, (pages 1 to 48). The type used is good and clear, and the lines are not crowded, (being only 26 in a page).

The "Canon" or Power-Table above described is a remarkable Table for its epoch (1650—60), and is probably even now *unique* in the extent of its Sixth Power-Table, (1^6 to 200^6). The numbers—some of which contain 14 figures—are printed *continuously*, not broken into periods; so they are difficult to read. Both books are now rare, and so difficult to obtain as to be only of historic interest: it has therefore not been thought worth while to verify these Tables.

2. Dodson, James. *The Calculator, being correct and necessary Tables for computation, &c.*; London, 1747.

Tab. XXI., p. 42, gives the values of 2^n, from $n=0$ to $n=29$.

Tab. XXII., p. 44, gives the values of a^n, from $a=1$ to 9, $n=1$ to 20. These two Tables have been examined *throughout*. *No* errors found.

3. Hill, John. *Arithmetic both in the theory and practice, &c.*; *Edinbro'*, 1764.

The Table, pp. 159 to 161, styled a *Table of geometrical progression, &c*, gives the value of 2^{m-1}, to Argument m, from $m=1$ to 144: and, on page 156, the last line of a worked problem gives the valne of $(2^{144}-1)$: so that this Table gives in effect the values of 2^n from $n=0$ to 144, (the Argument being $m=n-1$). The type is clear, but very small; the numbers (the largest has 43 figures) are printed *continuously* (without breaks), so are difficult to read.

This Table has been collated *throughout* with* Mr. Woodall's MS., and also with several printed Tables up to their various limits. *No* errors found.

4. Lambert, J. H. *Zusätze zu den logarithmischen und trigonometrischen Tabellen, &c.*; Berlin, 1770.

Tab. VII., p. 118, styled *Dignitates Binarii* gives 2^n up to $n=70$.

Tab. VIII., p. 119, styled *Dignitates Ternarii* gives 3^n up to $n=50$.

Tab. IX., p. 120, styled *Dignitates Quinarii* gives 5^n up to $n=50$.

Tab. XL., pp. 202 to 206, styled *Dignitates partium unitatis centesimalium* gives the values (in decimals, but omitting the decimal point) of x^n, when $x=a\div100$, from $R=1$ to 100, $n=1$ to 11. The values are *exact* up to $n=4$, but only approximate (only 8 decimal places being printed) beyond $n=4$; hence this Table can be used as an ordinary power-table (of a^n) by simply omitting the unnecessary ciphers on the left—up to the 4th power, (but not beyond).

Tab. VII. was compared with Hill's Table *throughout*. *No* errors found.

Tab. VIII. was compared with Vega's, Kulik's, and Woodall's (MS.) Tables up to their respective limits (3^{30}, 3^{37}, 3^{40}), and with Glaisher's Table up to 3^{50}; the powers not printed in those Tables (41, 43, 46, 47, 49) were specially recomputed. *No* errors found.

Tab. IX. was compared with Callet's Table throughout. *No* errors found.

Tab. XL. was compared with Barlow's Table (up to the 4th power only). No errors were found in Tab. VII., VIII., IX. *Two* errors were found in Tab. XL., affecting the final digits only, viz.

In 36^3, Read .. 656 (not .. 659); In 74^4, Read .. 576 (not .. 571).

5. Schulze, J. C. *Neue und erweiterte Sammlung logarithmischer, trigonometrischer, &c. .... Tabellen*, Berlin, 1778.

The Table, pp. 278 to 281, styled *Tafel der Potenzen aller Wurzeln, so zwischen* 0.01 *und* 1.00 *fallen*, gives the values (in decimals) of x^n, where $x=a\div100$, from $a=1$ to 100, $n=1$ to 11. This Table is precisely like Lambert's Tab. XL., (except that the decimals are preceded by a cipher, and the decimal point is marked by a *comma*), and may be used as an

* This was kindly done by Mr. Woodall himself.

ordinary power-table (of a^n) up to the 4th power, (but not beyond) by simply omitting the comma which marks the decimal point, and the unnecessary ciphers on the left. It has been examined so far as the fourth power, and was found to contain *two* Errata, (*same* two as in Lambert's Table XL.), viz.

In $0,36^{[illegible]}$, Read .. 656, (not .. 6[illegible]9); In $0,71^4$, Read .. 576, (not .. 571).

6. HUTTON, CHAS. *Tables of the Products and Powers of Numbers, &c.*, London, 1781.

The Table, p. 101, gives a^n, from $a = 1$ to 99, $n = 1$ to 10, (all in one page). This book is an inconveniently large folio, (the pages being $16\frac{3}{4}$ in. by $10\frac{1}{2}$ in.); the type is good and clear, but is too crowded to be easily read; the numbers, many of which contain 20 figures, are printed *continuously*, not broken into periods, and are therefore difficult to read.

This Table has been compared throughout with Barlow's and Glaisher's Tables. *Three* errors were found.

Extent of the Table (in the Title); Read $a = 1$ to 99, (not 1 to 100).

In last 3 figures* of 81^5, Read .. 4401, (not .. 4101);

of $98^{[illegible]}$, Read .. 672, (not .. 662).

7. CALLET, FR. *Tables portatives de Logarithmes, &c.*; Ed. Ster., Paris, 1795 [1829].

The Table, p. 13, styled *Logarithmes ou Exposants des puissances fractionnaires de* 10, gives the (accurate) values expressed in decimals—of $(0.5)^n$ from $n = 1$ to 60, and may therefore be used as a power-table of 5^n, $n = 1$ to 60, (by simply omitting the decimal point and all the ciphers on the left). The Argument (n) is (most inconveniently) denoted by the letters of the alphabet (omitting j and w): thus a to z denote $n = 1$ to 24, a′ to z′ denote 25 to 48; a″ to m″ denote 49 to 60.

Callet's Tables suffer badly from want of a Table of Contents and Index: the existence of this exceptionally long power-table of 5^n could not be suspected. Attention was drawn to it by Dr. Glaisher (Brit. Assoc. Report above quoted, p. 92).

The Table was compared with Lambert's Table of 5^n up to $n = 50$, and from $n = 51$ to 60 by recomputing. *No* errors found.

8. VEGA, GEORG. *Tabulæ logarithmico-trigonometricæ cum diversis aliis, &c.*, Lipsiæ, Vol. II., 1797.

The Tables following Table II. give 2^n, 3^n, and 5^n from $n = 1$ to 45, 36, 27 respectively.

Tab. IV. gives a^n from $a = 1$ to 100, $n = 1$ to 9.

[These Tables are now only of historic interest: they have *not been examined*, as they are practically superseded by the later Edition by Hulsse, reported on below, No. 12.]

9. BARLOW, PETER. *New Mathematical Tables, &c.*, London, 1814.

Tab. II., pp. 170 to 173 gives a^n from $a = 1$ to 100, $n = 1$ to 10.

Tab. III., pp. 175 to 181 gives a^4 and a^5 from $a = 100$ to $a = 999$.

These Tables are very closely printed in small type, and the type is not very clear: the numbers, many of which contain 20 figures, are printed *continuously*, not broken into periods, and are therefore difficult to read.

Tab. II. has been compared throughout with Hutton's and Glaisher's Tables.

Tab. III. has been compared throughout with Glaisher's Tables.

Errata were found as follows:—Tab. II., 2 in a^8, 2 in a^{10}; Tab. III., 11 in a^4, 27 in a^5.

Corrigenda in Tab. II.

Number;	.... 52^8	100^9	17^{10}	100^{10}
Read;	 712	18 ciphers	 149	20 ciphers
Not;	 172	16 ciphers	 119	19 ciphers

* These two Errata had been previously noted by Dr Glaisher (Brit. Assoc. Report above quoted, p. [illegible]).

a	Fourth Power Read	Fourth Power Not	a	Fifth Power Read	Fifth Power Not	a	Fifth Power Read	Fifth Power Not
402	2611585...	2611495...	403	...243	...253	767	...44638774 8607	...40577675 8607
403	...6683281	...6233281	405	...125	...135	777	283207724...	283211954...
415	2966145...	2965215...	483	...8667488264 3	...8766786947 3	805	3380488025...	3380401085...
525	...140625	...185695	507	3349961...	3349931...	806	...91776	92096
526	...608976	...698976	511	...63551	...63731	807	...69084820807	...66179620807
527	3397441	3442441	533	...596437893	...546437943	827	...836591312907	...841681802907
607	1357546...	1357516...	555	...7346875	...7297375	877	...148157	...146357
629	...0881	...0431	559	5458320...	5458770...	899	...904499	...906299
707	...022801	...015701	561	...61698801	...62240331	915	...16071875	...15871875
804	41785364...	41783224...	569	...4333849	...4233849	965	...828700603125	...844588392405
805	41993640...	41992560...	595	74573551...	74574451...	966	...1173596244576	...1205371823136
806	42202693...	42201973...	725	...9453125	...9953125	967	...53652046960 7	...52408258887
807	42412526...	42412166...	751	...128751	...125151	983	...185143	...183343
999	996005...	996095...	763	2585965...	2586055...			

10. Kulik, J. P. *Divisores Numerorum decies centena millia, &c.* Graecii, 1825.

Tab. 7, p. 286, styled *Potentiæ altiores binarii, ternarii, et quinquenarii,* gives 2^n from $n=10$ to 71; 3^n and 5^n from $n=7$ to 37.

These Tables are arranged in such a way as to save a good deal of space; thus the values 2^{10} to 2^{40} run *down* and those of 2^{41} to 2^{70} run *up* the page, the large values of the latter being ranged in line with and *close up to* the small values of the former. Similarly the values of 3^n run *down* the page, ranged with, and close to those of 5^n which run *up* the page.

These Tables have been compared *throughout* with Hill's and Lambert's Tables. Only *one* error was found, viz.

2^{22} should end with ....304, (not ....303).

11. Jahn, G. Ad. *Tafeln der Quadrat-und Kubikwurzeln aller Zahlen von* 1 *bis* 25500, *&c.*, Ster. Ed., Leipzig, 1839.

Tab. I. (of the Appendix), p. 241, gives a^n from $a=1$ to 50, $n=4$ to 9.

The type used in this Table is clear, but the numbers—many of which contain 15 or 16 figures—are printed *continuously*, not broken into periods, so are difficult to read.

The Table has been compared *throughout* with Barlow's and Glaisher's Tables.

For so short a Table (about $\frac{3}{4}$ of a page), quite an unusual number of Errata were found, viz. 25, as below.

a^n	5th, 6th, & 7th Powers		a	8th Powers		a	9th Powers	
	Read	Not		Read	Not		Read	Not
17^5	...9857	...9357	13	...30721	...33321	13	...4499373	...45331
4^6	4096	4066	17	...7441	...7101	17	...76497	...7071
13^7	...8517	...8717	27	2824295...	2822675...	27	76255974...	76212234...
17^7	...8673	...8653	28	...01998336	...02000576	28	...55953408	...5601612
27^7	104603...	104543...	32	...776	...746	32	...88832	...8787
28^7	...512	...592	37	...9453921	...9527921	37	...739735077	...7425330
37^7	...77133	...79133	42	9682651...	9683971...	40	2621440...	2621360...
42^7	230539...	230549...	48	2817...	2827...	42	406671383...	406689023...
						48	135260...	135740...

12. Vega, Georg F. von. *Sammlung mathematischer Tafeln.* Ed. J. A. Hülsse; Ster. Ed. 1840, [2nd impression, 1849.]

After Tab. VI., p. 466, follow three short Tables giving 2^n from $n=1$ to 45; 3^n from $n=1$ to 36; 5^n from $n=1$ to 27.

Tab. IXa, pp. 578 to 581, styled *Potenzen-Tafel,* gives the values (in decimals) of x^n, where $x=a\div 100$, from $a=1$ to 100, $n=1$ to 11. The values are *exact* up to $n=4$, but only approximate (only 8 decimal places being printed) beyond $n=4$; hence this Table may be used as an ordinary power-table (of a^n) by simply omitting the decimal point and the unnecessary ciphers on the left—up to the 4th power (but not beyond).

Tab. IXb, pp. 582 to 585, styled *Potenzen-Tafel,* gives a^n from $a=1$ to 100, $n=1$ to 9. The type of these Tables is bold and clear, but the numbers in Tab. IXb (many of which contain 18 figures) are printed *continuously* without any break, so require care in reading.

The Tables of 2^n, 3^n, 5^n have been compared *throughout* with Hill's and Lambert's Tables: *no* errors were found. Tab. IXa was compared (as far

as the fourth power only), with Barlow's Table III.: *two* errata* were found (see below). Tab. IXB was compared *throughout* with Barlow's Table II.; *no* errors found.

Corrigenda in Tab. IXA. In 0.36^3 read ....656 (not ....659);
in 0.74^4 read ....576 (not ....571).

13. KÖHLER, H. G. *Logarithmisch-trigonometrisches Handbuch, &c.* ster. Ed., Leipzig, 1848 [9th ster. Ed., 1864].

Tab. II., p. 355, gives 2^n from $n=1$ to 45, 3^n from $n=1$ to 36, 5^n from $n=1$ to 27.

Tab IV., pp. 360 to 363 gives a^n, from $a=1$ to 100, $n=1$ to 9.

Tab. II. was compared *throughout* with Hill's and Lambert's Tables. Tab. IV. was compared *throughout* with Barlow's Table II. *No* errors found.

14. SHANKS, WILLIAM. *Contributions to Mathematics, comprising chiefly the Rectification of the Circle to* 607 *places of decimals.* London, 1853.

The Table, pp. 90 to 95, styled *Powers of the Number* 2, gives 2^n, where $n=12\nu+1$, from $n=13$ to 721.

This Table has been compared with Hill's Table up to $n=145$; and 2^{253}, 2^{517}, 2^{721} have been compared with Martin's 2^{256}, and 2^{512}, and Putnam's 2^{721}, (see Tab. Nos. 19, 23 *below*). *No* errors found.

15. BEARDMORE, NATHL. *Manual of Hydrology, containing &c.* London, 1862.

Tab. **35**, pp. 86, 87 gives a^2, a^3, and a^5 from $a=1$ to 100.

This Table was compared *throughout* with Barlow's Tab. II. *One* error was found, viz., 100^3 should have *six* ciphers (only 5 printed).

16. MOLESWORTH, G. L. *Pocket-book of Useful Formulæ and Memoranda for Civil and Mechanical Engineers.* London, 1862. [21st Ed., 1882].

The Table, p. 709, gives a^4 and a^5 from $a=1$ to 100.

This Table was compared *throughout* with Barlow's Tab. II. *Two* errors were found, viz.,

In 86^4 read ...00816 (not ....08016); in 70^5 read 16807 ....(not 16847....)

[The author of this Work reports now that these *two* errors run through *all* the Editions, from the 1st to the most recent (25th) Edition].

17. RANKINE, WM. J. M. *Useful Rules and Tables.* London, 1866. [2nd Ed., 1867].

The Table, p. 32, gives a^2 and a^5 from $a=10$ to 99.

This Table was compared *throughout* with Barlow's Tab. II. *One* error was found, viz.,

In units digit of 82^5 read ....2 (not ....8).

18. HOÜEL, J. *Recueil de Formules et de Tables Numériques.* Paris, 1866.

Tab. XIX., p. 64, styled *Tables des Puissances*, contains several short Tables.

[3]. Table giving 2^n from $n=1$ to 45.

[4]. Table giving a^n: $a=3, 5, 6, 7, 9$; 11, 12, 13, 14. 15; $n=1$ to 10.

The Table of 2^n was compared *throughout* with Lambert's Table. The Table of a^n was compared *throughout* with Barlow's Tab. II. *One* error was found, viz.,

In 12^8 read 429.... (not 439 ....).

19. MARTIN, ARTEMAS, in the *Mathematical Visitor*, Vol. II., 1883, p. 32.

The solution of Question 254 gives 2^n, $n=2^\nu$ from $\nu=1$ to 9.

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* Tab. IXA is precisely the *same* as Lambert's Table XL. and Schulze's Table on pp. 278–281, and contains the *same* two errata.

Solutions by Messrs. K. S. Putnam (Rome, N.Y.), L. Brown (Hudson, Mass.); A. R. Bullis (Macedon, N.Y.)

The results were compared with Hill's Table from $\nu=1$ to 7; the values for $\nu=8$ and 9 were compared (by special computing) with Shanks's Tables. *No* errors found.

20. ADAMS, J. C. *Cambridge Observations for* 1866-7-8-9, Vol. XXII., Cambridge, 1890.

Tab. IV. (in Appendix I., p. xiii.) gives 2^n, n odd from $n=1$ to 125.

This Table was compared *throughout* with Hill's Table. *No* errors found.

21. LAFFAILLE, J. *La Science des Chiffres, &c.*, Montrouge-Seine, 1896.

The Table, p. 62, gives a^n from $a=1$ to 9, $n=1$ to 11.

This Table was compared *throughout* with Dodson's Table. *Two* errors were found, viz.,

In 4^{10} read **1048** ....(not **953** ....); in 4^{11} read **4194** ....(not **3834** ....).

22. WOODALL, H. J. }
23. PUTNAM, K. S. } Solution of Question 4329 in *Mathematical Questions and Solutions from the Educational Times*, Vol. LXVIII., 1898, p. 27.

2^{103} is given (by Mr. Woodall). Compared with Hill's Table. *No* errors found.

2^{721} is given (by Mr. Putnam). Compared with Shanks's Table. *No* errors found.

2^{1024} is given (by Mr. Putnam). This is stated to have been obtained by squaring the value of 2^{512} given in No. 19 above quoted; by dividing this by 2^{303} a *new* value of 2^{721} was obtained. Compared now with Shanks's value of 2^{721}. *No* errors found.

[The above suffices to confirm the *printed* values of 2^{103}, 2^{512} (in No. 19 above), and 2^{721}, and the MS. (but *not the printed*) value of 2^{1024}.

24. CUNNINGHAM, ALLAN, J. C., *Lt.-Col. R.E.* *Binary Canon*. London, 1900.

The Table on p. viii. of the Introduction gives 2^n from $n=1$ to 45.

This Table was collated with Vega's Table. *No* errors found.

25. GLAISHER, J. W. L., DR. [*Power-Tables*, 1873? Unpub.]

These give a^n, from $a=0$ to 999, $n=1$ to 12: and thus also give virtually powers (n) higher than the 12th of the following small numbers (when the exponent n is not a prime)

$$2^{108},\ 3^{72},\ 4^{54},\ 5^{48};\ \ 6^{36},\ 7^{36},\ 8^{36},\ 9^{36};\ \ 11^{24} \text{ to } 31^{24}.$$

These fine Tables are believed to be the largest Power-Tables in existence. They are beautifully printed in clear type, with the numbers broken into periods, so as to be easily read. They were reported in the British Association Report of 1873 (drawn up by Mr. Glaisher himself) as being then ready in duplicate: they have since then been completed and stereotyped, but are not yet published; the delay in publication is a great loss to arithmetical science.

[A single copy of these Tables was kindly lent to the author of this Paper some years ago, and has proved of great use in preparing this Paper, and in much other work. The Tables appear to be very accurate. All the discrepancies between this and the other Tables reported on have turned out to be due to errata in the other Tables.]

26. CUNNINGHAM, ALLAN J. C., *Lt.-Col. R.E.*, 1875. Tables in MS. only.

These Tables give 2^n, 3^n, 4^n, 5^n from $n=1$ to 32; 6^n, 7^n, 8^n, 9^n from $n=1$ to 20; 11^n to 25^n from $n=1$ to 16; 26^n and 27^n from $n=1$ to 10.

These Tables have proved useful in collating with several of the printed Tables. *No* errors found.

27. Woodall, H. J., 1898 to 1901. Tables (in MS. only) giving 2^n from $n=1$ to 144; and 3^n and 5^n from $n=1$ to 40.

[Mr. Woodall has kindly collated Hill's Table of 2^n with this Table, and also lent the Tables of 3^n, 5^n for collation. *No* errors found.]

Appendix II. Tables of Squares and Cubes. Errata.

Here follow a few notes of Errata in Tables of Squares and Cubes contained in the works reported on above as Nos. 9, 11, and in two other books of Tables.

9. Barlow, P., *New Mathematical Tables, &c.* London, 1814.
Tab. I., pp. 1 to 167 gives a^2 and a^3 from $a=1$ to 10000.
Errata.—A list of 14 Errata in the Squares and 33 Errata in the Cubes is given in *Barlow's Tables of Squares, Cubes, &c*, ster. Ed., London, 1886; Preface, p. 5. This later work gives a *corrected* edition of the Table of Squares and Cubes to the same extent (up to 10000) as the former, and is believed to be very correct. As the old (1814) Edition is now difficult to obtain, and is practically superseded by the later one, it seems sufficient to quote the above reference to the List of the Errata without detailing them.

11. Jahn, G. Ad. *Tafeln der Quadrat-und Kubikwurzeln aller Zahlen von* 1 *bis* 25500, &c, ster. Ed., Leipzig, 1839.
Tab. II., pp. 97 to 157 gives a^2 from $a=0$ to 27000.
Tab. III., pp. 161 to 237 gives a^3 from $a=0$ to 24000.

a	Corrig. in Squares		a	Corrig. in Squares	
	Read	Not		Read	Not
1188	1411...	1441...	10613	...5769	...5669
2394	...236	...239	11390	...2100	...2400
2526	...676	...626	11455	...17025	...10025
2527	...729	...789	11833	...019889	...009889
2621	68696...	68666...	12139	...55321	...58321
2973	...729	...720	13340	...5801	...5861
5992	4064	...4664	14298	...2804	...2104
9310	6100	.. 6010	14789	...14521	...18521
			14827	...9929	...9529

Corrigenda in Cubes.

10525^3; Read....3125, Not....3225

22458^3; Read....56495.., Not....55495..

The type used in these Tables is clear; but the numbers, many of which contain 14 figures, are printed *continuously*, not broken into periods. This is *one of the largest* Tables of Squares and Cubes in existence, very few Tables extending beyond 10000^2, and still fewer beyond 10000^3. Unfortunately the numerous errors (25) found in one page (p. 241) of the Power-Tables (reported in No. 11 above) shake one's confidence in the main Table (of Squares and Cubes).

Kulik, J. P. Tafeln der Quadrat-und Kubik-zahlen aller natürlichen *Zahlen bis hundert-tausend, &c.*; Leipzig, 1848.

page 246, Head-line; For 506; 606; 706; 806; 906;
Read 560; 660; 760; 860; 960;

page 349, Col. of N; change all the Arguments (N).

For 00....09, 10....19, 20....29, 30....39, 40....49;
Read 50....59, 60....69, 70....79, 80....89, 90....99;

Mathematical Tables, &c., [Anon.], pub. at the Thomason C. E. Coll. Press, Roorkee, N.W.P., India; 1st Edn. n.d.; 3rd Edn. 1898.

Errata in the Tables of Squares, Cubes, &c., [in some Editions].

In 454^2, Read....106 (not....116;
In 484^2, Read....256 (not....255);
In 611^2, Read 373.... (not 173....);
In 626^3, Read 245314.... (not 245134....).

Addenda, &c., to the Author's Paper *On Factor-Tables*, *Errata* in Vol. xxxiv. of this Journal.

page 26.

(1). Chernac's *Cribrum Arithmeticum.*

Additional Corrigenda in column of Divisors (D); [Argument N].

Number (N),	Divisors (D);	Number (N),	Divisors (D);
469273,	7.7.61.157;	494543,	7.31.43.53;

page 29.

(6). Goldberg's *Primzahlen und Factoren-Tafel, &c.*

(1). *Additional Corrigenda in the Number-Column.*

Page	Line	For	Read;	Page	Line	For	Read;
123	29	103373	108373;	239	10	240733	210733;

(2). *Additional Corrigenda in the Factors-Column*, [Argument N].

Number	Factors
79679	17.43.109.

Corrigenda on the same Paper.

page 30; Table (6)–(2). In Number-Column; For *53293, Read 53293.

Corrigenda in Factor-Column (in some copies only), [Argument N].

page 30; For 94987 = 43.47, Read = 43.47^2.
page 31; For 165997 = 13.113, Read = 13.113^2.

ON THE ELLIPTIC AND ZETA FUNCTIONS OF $\frac{2}{3}K$.

By *J. W. L. Glaisher.*

Introduction, §§ 1–6.

§ 1. In Vol. XII. of the *Messenger** Professor Burnside gave the values of $\mathrm{sn}^4\frac{4}{3}K$, $\mathrm{sn}^4\frac{4}{3}iK'$, &c., in terms of k. His results lead directly to the formulæ

$$k^2\,\mathrm{sn}^2\tfrac{2}{3}K = kL,$$

$$k^2k'^2\,\mathrm{sd}^2\tfrac{2}{3}K = kk'M,$$

$$k'^2\,\mathrm{sc}^2\tfrac{2}{3}K = k'N,$$

where
$$L = \sqrt{\{2(1-\lambda+\lambda^2)^{\frac{1}{2}} + 2 - \lambda\}} - \sqrt{(1+\lambda)},$$

$$M = \sqrt{\{2(1+\mu+\mu^2)^{\frac{1}{2}} - 2 - \mu\}} \mp \sqrt{(\mu-1)},$$

$$N = \sqrt{\{2(1-\nu+\nu^2)^{\frac{1}{2}} + 2 - \nu\}} + \sqrt{(1+\nu)},$$

and
$$\lambda = \left(\frac{k'^2}{2k}\right)^{\frac{2}{3}}, \quad \mu = \left(\frac{1}{2kk'}\right)^{\frac{2}{3}}, \quad \nu = \left(\frac{k^2}{2k'}\right)^{\frac{2}{3}}.$$

In M the upper or lower sign is to be taken according as $k <$ or $> k'$.

§ 2. In a paper† in Vol. XXI. (1890) of the *Proceedings of the London Mathematical Society* I deduced from these values of Burnside's the following formulæ:

$$\mathrm{cd}\,\tfrac{2}{3}K = \tfrac{1}{2}(1+U), \qquad \mathrm{cn}\,\tfrac{2}{3}K = \tfrac{1}{2}(1-U),$$

$$\mathrm{dc}\,\tfrac{2}{3}K = \tfrac{1}{2}(1+W), \qquad \mathrm{dn}\,\tfrac{2}{3}K = \tfrac{1}{2}(-1+W),$$

$$\mathrm{nc}\,\tfrac{2}{3}K = \tfrac{1}{2}(1+V), \qquad \mathrm{nd}\,\tfrac{2}{3}K = \tfrac{1}{2}(-1+V),$$

where
$$U = \sqrt{\{2(1-u+u^2)^{\frac{1}{2}} + 2 - u\}} - \sqrt{(1+u)},$$

$$W = \sqrt{\{2(1+w+w^2)^{\frac{1}{2}} + 2 + w\}} \pm \sqrt{(1-w)},$$

$$V = \sqrt{\{2(1-v+v^2)^{\frac{1}{2}} + 2 - v\}} + \sqrt{(1+v)},$$

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* "The elliptic functions of $\frac{1}{3}K$," pp. 154—157.

† "On the q-series derived from the elliptic and Zeta functions of $\frac{1}{3}K$ and $\frac{1}{4}K$," pp. 143—171. In this paper the argument $\frac{1}{3}K$ was used, but I have found that when the argument is $\frac{2}{3}K$ the formulæ are much more regular. In the results quoted from this paper the argument has been changed from $\frac{1}{3}K$ to $\frac{2}{3}K$ by making the corresponding change in the function.

The formulæ for $\mathrm{cd}\,\frac{2}{3}K$, &c., in terms of U, &c., were also given (with argument $\frac{1}{3}K$) in the *Messenger*, Vol. XX., pp. 191—192.

and $$u=\left(\frac{2k'}{k^2}\right)^{\frac{2}{3}},\quad w=(2kk')^{\frac{2}{3}},\quad v=\left(\frac{2k}{k'^2}\right)^{\frac{2}{3}}.$$

In W the upper or lower sign is to be taken according as $k <$ or $> k'$.

From these formulæ, by addition and subtraction, we find

$$\begin{aligned}
&\mathrm{cd}\,\tfrac{2}{3}K-\mathrm{cn}\,\tfrac{2}{3}K=U, &&\mathrm{cd}\,\tfrac{2}{3}K+\mathrm{cn}\,\tfrac{2}{3}K=1,\\
&\mathrm{dc}\,\tfrac{2}{3}K+\mathrm{dn}\,\tfrac{2}{3}K=W, &&\mathrm{dc}\,\tfrac{2}{3}K-\mathrm{dn}\,\tfrac{2}{3}K=1,\\
&\mathrm{nc}\,\tfrac{2}{3}K+\mathrm{nd}\,\tfrac{2}{3}K=V, &&\mathrm{nc}\,\tfrac{2}{3}K-\mathrm{nd}\,\tfrac{2}{3}K=1;
\end{aligned}$$

whence, or otherwise, it can be shown that

$$\frac{k\,\mathrm{sn}\,\tfrac{2}{3}K\,\mathrm{cn}\,\tfrac{2}{3}K}{\mathrm{dn}\,\tfrac{2}{3}K}=U^{\frac{1}{2}},$$

$$\frac{\mathrm{sn}\,\tfrac{2}{3}K\,\mathrm{dn}\,\tfrac{2}{3}K}{\mathrm{cn}\,\tfrac{2}{3}K}=W^{\frac{1}{2}},$$

$$\frac{k'\,\mathrm{sn}\,\tfrac{2}{3}K}{\mathrm{cn}\,\tfrac{2}{3}K\,\mathrm{dn}\,\tfrac{2}{3}K}=V^{\frac{1}{2}}.$$

§3. The formulæ of the preceding section express the six even elliptic functions of $\tfrac{2}{3}K$ in terms of U, V, W. Burnside's formulæ express three of the uneven elliptic functions in terms of $L^{\frac{1}{2}}$, $M^{\frac{1}{2}}$, $N^{\frac{1}{2}}$ respectively, viz., the formulæ are

$$\mathrm{sn}\,\tfrac{2}{3}K=\frac{L^{\frac{1}{2}}}{k^{\frac{1}{2}}},\quad \mathrm{sd}\,\tfrac{2}{3}K=\frac{M^{\frac{1}{2}}}{k^{\frac{1}{2}}k'^{\frac{1}{2}}},\quad \mathrm{sc}\,\tfrac{2}{3}K=\frac{N^{\frac{1}{2}}}{k'^{\frac{1}{2}}}.$$

The values of the other three uneven elliptic functions, derived from these equations by taking their reciprocals, are

$$\mathrm{ns}\,\tfrac{2}{3}K=\frac{k^{\frac{1}{2}}}{L^{\frac{1}{2}}},\quad \mathrm{ds}\,\tfrac{2}{3}K=\frac{k^{\frac{1}{2}}k'^{\frac{1}{2}}}{M^{\frac{1}{2}}},\quad \mathrm{cs}\,\tfrac{2}{3}K=\frac{k'^{\frac{1}{2}}}{N^{\frac{1}{2}}}.$$

These latter formulæ, however, are less satisfactory, as the complicated radical is in the denominator.

§4. In a paper which I communicated last year to the *London Mathematical Society** I have given the expansions in powers of q of the sixteen elliptic and four Zeta functions in a form in which the coefficients are expressed by means of only two arithmetical functions $E(n)$ and $H(n)$. These formulæ afford expressions for $\mathrm{ns}\,\tfrac{2}{3}K$, $\mathrm{ds}\,\tfrac{2}{3}K$, $\mathrm{cs}\,\tfrac{2}{3}K$ as linear functions of $U^{\frac{1}{2}}$, $V^{\frac{1}{2}}$, $W^{\frac{1}{2}}$ and another radical of the same kind.

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* *Proceedings*, Ser. II., Vol. II., pp. 310—351

The q-series which occur in the expansions may also be expressed in the same manner. These formulæ will now be investigated.

§ 5. The sixteen expansions which have just been referred to are:

$$k\rho \operatorname{cd} \tfrac{2}{3}K = 2\Sigma_1^\infty \{E(m) + 3E(3m)\} q^{\frac{1}{2}m},$$

$$k\rho \operatorname{cn} \tfrac{2}{3}K = 2\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} \{E(m) + 3E(3m)\} q^{\frac{1}{2}m},$$

$$\rho \operatorname{dn} \tfrac{2}{3}K = 1 - 2\Sigma_1^\infty \{E(n) - 3E(3n)\} q^n,$$

$$k'\rho \operatorname{nd} \tfrac{2}{3}K = 1 - 2\Sigma_1^\infty (-1)^n \{E(n) - 3E(3n)\} q^n,$$

$$\rho \operatorname{dc} \tfrac{2}{3}K = 2 + 2\Sigma_1^\infty \{E(n) + 3E(3n)\} q^n,$$

$$k'\rho \operatorname{nc} \tfrac{2}{3}K = 2 + 2\Sigma_1^\infty (-1)^n \{E(n) + 3E(3n)\} q^n,$$

$$k\rho \operatorname{sn} \tfrac{2}{3}K = 2\sqrt{3}\, \Sigma_1^\infty H(m) q^{\frac{1}{2}m},$$

$$kk'\rho \operatorname{sd} \tfrac{2}{3}K = 2\sqrt{3}\, \Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} H(m) q^{\frac{1}{2}m},$$

$$\rho \operatorname{zn} \tfrac{2}{3}K = 2\sqrt{3}\, \Sigma_1^\infty \{H(n) - H(2n)\} q^n,$$

$$\rho \operatorname{zd} \tfrac{2}{3}K = 2\sqrt{3}\, \Sigma_1^\infty (-1)^n \{H(n) - H(2n)\} q^n,$$

$$\rho \operatorname{ns} \tfrac{2}{3}K = \frac{2}{\sqrt{3}} + 2\sqrt{3}\, \Sigma_1^\infty \{H(n) + H(2n)\} q^n,$$

$$\rho \operatorname{ds} \tfrac{2}{3}K = \frac{2}{\sqrt{3}} + 2\sqrt{3}\, \Sigma_1^\infty (-1)^n \{H(n) + H(2n)\} q^n,$$

$$\rho \operatorname{zs} \tfrac{2}{3}K = \frac{1}{\sqrt{3}} + 2\sqrt{3}\, \Sigma_1^\infty H(n) q^{2n},$$

$$\rho \operatorname{cs} \tfrac{2}{3}K = \frac{1}{\sqrt{3}} - 2\sqrt{3}\, \Sigma_1^\infty \{H(n) - 2H(2n)\} q^{2n},$$

$$\rho \operatorname{zc} \tfrac{2}{3}K = -\sqrt{3} - 2\sqrt{3}\, \Sigma_1^\infty \{H(n) + 2H(2n)\} q^{2n},$$

$$k'\rho \operatorname{sc} \tfrac{2}{3}K = \sqrt{3} + 2\sqrt{3}\, \Sigma_1^\infty \{1 + 2(-1)^n\} H(n) q^{2n},$$

where $\rho = \dfrac{2K}{\pi}$, n is any number, m any uneven number, $E(n)$ denotes the excess of the number of divisors of n of the form $4k+1$ over the number of those of the form $4k+3$, and $H(n)$ the excess of the number of divisors of n of the form $3k+1$ over the number of those of the form $3k+2$.*

* A table of the values of $E(n)$ up to $n = 1000$ was given in the *Proceedings of the London Mathematical Society*, Vol. XV. (1884), p. 106, and a table of $H(n)$ of the same extent in the *Messenger*, Vol. XXXI. (1901), pp. 64—72. Two errors in the former table are pointed out in the latter paper (p. 66).

§ 6. With the E-expressions I am not concerned further in this paper, as the principal formulæ derived from them (including the values of the even elliptic functions of $\frac{2}{3}K$) have been quoted in § 2. Other formulæ are given in the paper in Vol. XXI. of the *Proc. Lond. Math. Soc.*, which is there referred to.

The following values of q-series may, however, be noticed:

$$\Sigma_1^\infty E(3m)\,q^{\frac{1}{3}m} = \Sigma_1^\infty E(m)\,q^{\frac{2}{3}m} = \tfrac{1}{12}k^3\rho\,\frac{\operatorname{sn}^2\frac{2}{3}K\operatorname{cn}^2\frac{2}{3}K}{\operatorname{dn}^2\frac{2}{3}K} = \tfrac{1}{12}k\rho\,U,$$

$$1 + 4\Sigma_1^\infty E(3n)\,q^n = 1 + 4\Sigma_1^\infty E(n)\,q^{3n} = \tfrac{1}{3}\rho\,\frac{\operatorname{sn}^2\frac{2}{3}K\operatorname{dn}^2\frac{2}{3}K}{\operatorname{cn}^2\frac{2}{3}K} = \tfrac{1}{3}\rho\,W,$$

$$1 + 4\Sigma_1^\infty(-1)^n E(3n)\,q^n = 1 + 4\Sigma_1^\infty(-1)^n E(n)\,q^{3n}$$

$$= \tfrac{1}{3}k'^3\rho\,\frac{\operatorname{sn}^2\frac{2}{3}K}{\operatorname{cn}^2\frac{2}{3}K\operatorname{dn}^2\frac{2}{3}K} = \tfrac{1}{3}k'\rho\,V.$$

The equality of the two forms of q-series is easily seen; for taking, for example, the second formula, we have

$$\Sigma_1^\infty E(3n)\,q^n = \Sigma_1^\infty E(9n)\,q^{3n} + \Sigma_0^\infty E(9n+3)\,q^{3n+1}$$

$$+ \Sigma_0^\infty E(9n+6)\,q^{3n+2}.$$

Now $E(3^2.n) = E(n)$; and $E(9n+3)$ and $E(9n+6)$ are both zero, for in each case the argument is divisible by 3, but cannot be divisible by 3^2. Therefore

$$\Sigma_1^\infty E(3n)\,q^n = \Sigma_1^\infty E(n)\,q^{3n}.$$

The second form of q-series is the better, for in the first the coefficients of two-thirds of the terms are necessarily zero.

Values of elliptic and Zeta functions of $\frac{2}{3}K$ and of certain q-series, §§ 7–20.

§ 7. Passing now to the H-group we may write the expressions in the form

$$k\rho\operatorname{sn}\tfrac{2}{3}K = 2\sqrt{3}\,\Sigma_1^\infty H(m)\,q^{\frac{1}{3}m},$$

$$kk'\rho\operatorname{sd}\tfrac{2}{3}K = 2\sqrt{3}\,\Sigma_1^\infty(-1)^{\frac{1}{2}(m-1)}H(m)\,q^{\frac{1}{3}m},$$

$$\rho\operatorname{zn}\tfrac{2}{3}K = 2\sqrt{3}\,\Sigma_1^\infty H(n)\,q^n - 2\sqrt{3}\,\Sigma_1^\infty H(n)\,q^{3n},$$

$$\rho\operatorname{zd}\tfrac{2}{3}K = 2\sqrt{3}\,\Sigma_1^\infty(-1)^n H(n)\,q^n - 2\sqrt{3}\,\Sigma_1^\infty H(n)\,q^{3n},$$

$$\rho\operatorname{ns}\tfrac{2}{3}K = \frac{2}{\sqrt{3}} + 2\sqrt{3}\,\Sigma_1^\infty H(n)\,q^n + 2\sqrt{3}\,\Sigma_1^\infty H(n)\,q^{3n},$$

$$\rho \operatorname{ds} \tfrac{2}{3}K = \frac{2}{\sqrt{3}} + 2\sqrt{3}\Sigma_1^\infty (-1)^n H(n) q^n + 2\sqrt{3}\Sigma_1^\infty H(n) q^{3n},$$

$$\rho \operatorname{zs} \tfrac{2}{3}K = \frac{1}{\sqrt{3}} + 2\sqrt{3}\Sigma_1 H(n) q^{3n},$$

$$\rho \operatorname{cs} \tfrac{2}{3}K = \frac{1}{\sqrt{3}} - 2\sqrt{3}\Sigma_1^\infty H(n) q^{2n} + 4\sqrt{3}\Sigma_1^\infty H(n) q^{4n},$$

$$\rho \operatorname{zc} \tfrac{2}{3}K = -\sqrt{3} - 2\sqrt{3}\Sigma_1^\infty H(n) q^{2n} - 4\sqrt{3}\Sigma_1^\infty H(n) q^{4n},$$

$$k'\rho \operatorname{sc} \tfrac{2}{3}K = \sqrt{3} + 6\sqrt{3}\Sigma_1^\infty H(n) q^{2n} - 8\sqrt{3}\Sigma_1^\infty H(m) q^{2m}.$$

§ 8. In order therefore to express in terms of k the six uneven elliptic functions and the four Zeta functions, we require to know the values of

$$\Sigma_1^\infty H(m) q^{\frac{1}{2}m}, \quad \Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} H(m) q^{\frac{1}{2}m}, \quad \Sigma_1^\infty H(m) q^{2m},$$

$$1 + 6\Sigma_1^\infty H(n) q^n, \quad 1 + 6\Sigma_1^\infty (-1)^n H(n) q^n,$$

$$1 + 6\Sigma_1^\infty H(n) q^{2n}, \quad 1 + 6\Sigma_1^\infty H(n) q^{4n}.$$

It is to be observed that these seven quantities are not all independent, for

$$\begin{aligned}\Sigma_1^\infty H(n) q^n &= \Sigma_1^\infty H(2n) q^{2n} + \Sigma_1^\infty H(m) q^m \\ &= \Sigma_1^\infty H(4n) q^{4n} + \Sigma_1^\infty H(2m) q^{2m} + \Sigma_1^\infty H(m) q^m,\end{aligned}$$

and $H(4n) = H(2^2.n) = H(n)$, and $H(2m) = 0$, so that we have

$$(a) \qquad \Sigma_1^\infty H(n) q^n = \Sigma_1^\infty H(n) q^{4n} + \Sigma_1^\infty H(m) q^m.$$

Changing the sign of q in this equation, we find

$$\Sigma_1^\infty (-1)^n H(n) q^n = \Sigma_1^\infty H(n) q^{4n} - \Sigma_1^\infty H(m) q^m,$$

and therefore, by addition,

$$\begin{aligned}\{1 + 6\Sigma_1^\infty H(n) q^n\} + \{1 + 6\Sigma_1^\infty (-1)^n H(n) q^n\}& \\ = 2\{1 + 6\Sigma_1^\infty H(n) q^{4n}\}&.\end{aligned}$$

Writing q^2 for q in the formula (a), we find

$$1 + 6\Sigma_1^\infty H(n) q^{2n} = 1 + 6\Sigma_1^\infty H(n) q^{8n} + \Sigma_1^\infty H(m) q^{2m}.$$

The last formula in the group of expansions may therefore be written

$$k'\rho \operatorname{sc} \tfrac{2}{3}K = \sqrt{3} + 6\sqrt{3}\Sigma_1^\infty H(n) q^{8n} - 2\sqrt{3}\Sigma_1^\infty H(m) q^{2m}.$$

§ 9. Now Burnside's formulæ (§ 1) give the values of three of the uneven functions, viz.,

$$k \operatorname{sn} \tfrac{2}{3}K = k^{\frac{1}{2}}L^{\frac{1}{2}},$$

$$kk' \operatorname{sd} \tfrac{2}{3}K = k^{\frac{1}{2}}k'^{\frac{1}{2}}M^{\frac{1}{2}},$$

$$k' \operatorname{sc} \tfrac{2}{3}K = k'^{\frac{1}{2}}N^{\frac{1}{2}};$$

and therefore the formulæ from which we start are

$$\text{(i)}\quad 2\sqrt{3}\,\Sigma_1^\infty H(m)\,q^{\frac{1}{2}m} = k^{\frac{1}{2}}L^{\frac{1}{2}}\rho,$$

$$\text{(ii)}\quad 2\sqrt{3}\,\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)}H(m)\,q^{\frac{1}{2}m} = k^{\frac{1}{2}}k'^{\frac{1}{2}}M^{\frac{1}{2}}\rho,$$

$$\text{(iii)}\quad \sqrt{3}\,\{1 + 6\Sigma_1^\infty H(n)\,q^{2n}\} - 8\sqrt{3}\,\Sigma_1^\infty H(m)\,q^{2m}$$

$$= \sqrt{3}\,\{1 + 6\Sigma_1^\infty H(n)\,q^{8n}\} - 2\sqrt{3}\,\Sigma_1^\infty H(m)\,q^{2m} = k'^{\frac{1}{2}}N^{\frac{1}{2}}\rho.$$

In (i) change q into q^2; we thus find

$$\text{(iv)}\qquad 4\sqrt{3}\,\Sigma_1^\infty H(m)\,q^m = kU^{\frac{1}{2}}\rho.$$

Changing q and q^2 in (iv), we find

$$\text{(v)}\qquad 8\sqrt{3}\,\Sigma_1^\infty H(m)\,q^{2m} = (1-k')\,P^{\frac{1}{2}}\rho,$$

where $\qquad P = \sqrt{\{2(1-p+p^2) + 2 - p\}} - \sqrt{(1+p)},$

and $\qquad p = \dfrac{2^{\frac{4}{3}}k^{\frac{4}{3}}k'^{\frac{1}{3}}}{(1-k')^2}.$

Now change q into $q^{\frac{1}{2}}$ in (iii) : we thus find

$$\text{(viii)}\quad \sqrt{3}\,\{1 + 6\Sigma_1^\infty H(n)\,q^n - 8\sqrt{3}\,\Sigma_1^\infty H(m)\,q^m\}$$

$$= \sqrt{3}\,\{1 + 6\Sigma_1^\infty H(n)\,q^{4n} - 2\sqrt{3}\,\Sigma_1^\infty H(m)\,q^m\} = k'V^{\frac{1}{2}}\rho;$$

and, changing q into $q^{\frac{1}{2}}$ again,

$$\text{(ix)}\quad \sqrt{3}\,\{1 + 6\Sigma_1^\infty H(n)\,q^{2n} - 2\sqrt{3}\,\Sigma_1^\infty H(m)\,q^{\frac{1}{2}m}\} = (1-k)\,R^{\frac{1}{2}}\rho.$$

where $\qquad R = \sqrt{\{2(1-r+r^2) + 2 - r\}} + \sqrt{(1+r)},$

and $\qquad r = \dfrac{2^{\frac{4}{3}}k^{\frac{1}{3}}k'^{\frac{4}{3}}}{(1-k)^2}.$

§ 10. As the quantities $L^{\frac{1}{2}}$, $U^{\frac{1}{2}}$, ... occur only in connection with the factors $k^{\frac{1}{2}}$, k, ..., it is convenient to put

$$k^{\frac{1}{2}}L^{\frac{1}{2}} = L_0, \quad k^{\frac{1}{2}}k'^{\frac{1}{2}}M^{\frac{1}{2}} = M_0, \quad k'^{\frac{1}{2}}N^{\frac{1}{2}} = N_0,$$

$$kU^{\frac{1}{2}} = U_0, \quad k'V^{\frac{1}{2}} = V_0,$$

$$(1-k')\,P^{\frac{1}{2}} = P_0, \quad (1-k)\,R^{\frac{1}{2}} = R_0.$$

Using these abbreviations, the formulæ (i), ..., (ix) give

$$2\sqrt{3}\,\Sigma_1^\infty H(m)\,q^{\frac{1}{2}m} = L_0\rho,$$

$$2\sqrt{3}\,\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} H(m)\,q^{\frac{1}{2}m} = M_0\rho,$$

$$4\sqrt{3}\,\Sigma_1^\infty H(m)\,q^{m} = U_0\rho,$$

$$8\sqrt{3}\,\Sigma_1^\infty H(m)\,q^{2m} = P_0\rho,$$

$$\sqrt{3}\,\{1 + 6\Sigma_1^\infty H(n)\,q^n\} = (2U_0 + V_0)\,\rho,$$

$$\sqrt{3}\,\{1 + 6\Sigma_1^\infty (-1)^n H(n)\,q^n\} = (-U_0 + V_0)\,\rho,$$

$$\sqrt{3}\,\{1 + 6\Sigma_1^\infty H(n)\,q^{2n}\} = (N_0 + P_0)\,\rho = (L_0 + R_0)\,\rho,$$

$$\sqrt{3}\,\{1 + 6\Sigma_1^\infty H(n)\,q^{4n}\} = (\tfrac{1}{2}U_0 + V_0)\,\rho,$$

$$\sqrt{3}\,\{1 + 6\Sigma_1^\infty H(n)\,q^{8n}\} = (N_0 + \tfrac{1}{4}P_0)\,\rho.$$

§ 11. For brevity, let

$$a = \frac{1}{\sqrt{3}}\{1 + 6\Sigma_1^\infty H(n)\,q^n\},$$

$$b = \frac{1}{\sqrt{3}}\{1 + 6\Sigma_1^\infty (-1)^n H(n)\,q^n\},$$

$$c = \frac{1}{\sqrt{3}}\{1 + 6\Sigma_1^\infty H(n)\,q^{2n}\},$$

$$d = \frac{1}{\sqrt{3}}\{1 + 6\Sigma_1^\infty H(n)\,q^{4n}\},$$

$$e = 8\sqrt{3}\,\Sigma_1^\infty H(m)\,q^{2m},$$

so that (§ 8) $$a + b = 2d.$$

Then, from § 7, substituting for the series, we have

$$\rho\,\mathrm{zn}\,\tfrac{2}{3}K = a - c,$$

$$\rho\,\mathrm{zd}\,\tfrac{2}{3}K = b - c,$$

$$\rho\,\mathrm{ns}\,\tfrac{2}{3}K = a + c,$$

$$\rho\,\mathrm{ds}\,\tfrac{2}{3}K = b + c,$$

$$\rho\,\mathrm{zs}\,\tfrac{2}{3}K = c,$$

$$\rho\,\mathrm{cs}\,\tfrac{2}{3}K = 2d - c = a + b - c,$$

$$\rho\,\mathrm{zc}\,\tfrac{2}{3}K = -2d - c = -a - b - c,$$

$$k'\rho\,\mathrm{sc}\,\tfrac{2}{3}K = 3c - e.$$

§ 12. Before assigning to a, b, ... their values in terms of U_0, V_0, ... (§ 10) it is convenient to notice that a verification of these formulæ (and therefore also of the last eight q-expressions in § 7) may be obtained by means of the general formulæ:

$$\text{zs}\,2x = \tfrac{1}{2}(\text{zs}\,x + \text{zc}\,x + \text{zd}\,x + \text{zn}\,x),$$

$$\text{ns}\,2x = \tfrac{1}{2}(\text{zs}\,x - \text{zc}\,x - \text{zd}\,x + \text{zn}\,x),$$

$$\text{ds}\,2x = \tfrac{1}{2}(\text{zs}\,x - \text{zc}\,x + \text{zd}\,x - \text{zn}\,x),$$

$$\text{cs}\,2x = \tfrac{1}{2}(\text{zs}\,x + \text{zc}\,x - \text{zd}\,x - \text{zn}\,x)\,;$$

for putting $x = \frac{2}{3}K$, and observing that $\text{zs}\,\frac{4}{3}K = -\text{zs}\,\frac{2}{3}K$, $\text{ns}\,\frac{4}{3}K = \text{ns}\,\frac{2}{3}K$, &c., we find

$$-\text{zs}\,\tfrac{2}{3}K = \tfrac{1}{2}(\text{zs}\,\tfrac{2}{3}K + \text{zc}\,\tfrac{2}{3}K + \text{zd}\,\tfrac{2}{3}K + \text{zn}\,\tfrac{2}{3}K),$$

$$\text{ns}\,\tfrac{2}{3}K = \tfrac{1}{2}(\text{zs}\,\tfrac{2}{3}K - \text{zc}\,\tfrac{2}{3}K - \text{zd}\,\tfrac{2}{3}K + \text{zn}\,\tfrac{2}{3}K),$$

$$\text{ds}\,\tfrac{2}{3}K = \tfrac{1}{2}(\text{zs}\,\tfrac{2}{3}K - \text{zc}\,\tfrac{2}{3}K + \text{zd}\,\tfrac{2}{3}K - \text{zn}\,\tfrac{2}{3}K),$$

$$-\text{cs}\,\tfrac{2}{3}K = \tfrac{1}{2}(\text{zs}\,\tfrac{2}{3}K + \text{zc}\,\tfrac{2}{3}K - \text{zd}\,\tfrac{2}{3}K - \text{zn}\,\tfrac{2}{3}K).*$$

It is easy to verify that the values of the elliptic and Zeta functions given in the preceding section in terms of a, b, c satisfy these equations.

§ 13. From § 10 we have

$$a = \tfrac{1}{3}(2U_0 + V_0)\rho,$$

$$b = \tfrac{1}{3}(-U_0 + V_0)\rho,$$

$$c = \tfrac{1}{3}(N_0 + P_0)\rho\,;$$

and therefore, substituting these values for a, b, c,

$$\text{ns}\,\tfrac{2}{3}K = \tfrac{1}{3}(2U_0 + V_0 + N_0 + P_0),$$

$$\text{ds}\,\tfrac{2}{3}K = \tfrac{1}{3}(-U_0 + V_0 + N_0 + P_0),$$

$$\text{cs}\,\tfrac{2}{3}K = \tfrac{1}{3}(U_0 + 2V_0 - N_0 - P_0),$$

$$\text{zs}\,\tfrac{2}{3}K = \tfrac{1}{3}(N_0 + P_0),$$

$$\text{zc}\,\tfrac{2}{3}K = \tfrac{1}{3}(-U_0 - 2V_0 - N_0 - P_0),$$

$$\text{zd}\,\tfrac{2}{3}K = \tfrac{1}{3}(-U_0 + V_0 - N_0 - P_0),$$

$$\text{zn}\,\tfrac{2}{3}K = \tfrac{1}{3}(2U_0 + V_0 - N_0 - P_0)\,;$$

* We obtain the same formulæ by putting $x = \frac{1}{3}K$ and then changing the argument to $\frac{2}{3}K$.

which, with Burnside's original formulæ

$$k \operatorname{sn} \tfrac{2}{3}K = L_0, \quad kk' \operatorname{sd} \tfrac{2}{3}K = M_0, \quad k' \operatorname{sc} \tfrac{2}{3}K = N_0,$$

assign values to the ten uneven elliptic functions.

By these formulæ the s-group of elliptic functions and three of the Zeta functions of $\frac{2}{3}K$ are expressed as linear functions of the complicated radicals $U^{\frac{1}{2}}$, $V^{\frac{1}{2}}$, $N^{\frac{1}{2}}$, $P^{\frac{1}{2}}$. The expression for the other Zeta function involves only two radicals.

These expressions possess the advantage of being free from denominators, but it can scarcely be doubted that much simpler values must exist for the quantities in question.

In the seven formulæ N_0 and P_0 always occur in the same combination $N_0 + P_0$. We may in all cases replace $N_0 + P_0$ by $L_0 + R_0$, since these quantities are equal (§ 10).

§ 14. The quantity $W^{\frac{1}{2}}$ has not been used in the preceding investigation, although it might have been employed instead of $U^{\frac{1}{2}}$ or $V^{\frac{1}{2}}$ or in conjunction with them.

From § 2 we have

$$kU^{\frac{1}{2}} + k'V^{\frac{1}{2}} = \frac{k^2 \operatorname{sn} \frac{2}{3}K \operatorname{cn} \frac{2}{3}K}{\operatorname{dn} \frac{2}{3}K} + \frac{k'^2 \operatorname{sn} \frac{2}{3}K}{\operatorname{cn} \frac{2}{3}K \operatorname{dn} \frac{2}{3}K}$$

$$= \frac{\operatorname{sn} \frac{2}{3}K \operatorname{dn} \frac{2}{3}K}{\operatorname{cn} \frac{2}{3}K} = W^{\frac{1}{2}},$$

so that $$W^{\frac{1}{2}} = kU^{\frac{1}{2}} + k'V^{\frac{1}{2}},$$

or, putting $W^{\frac{1}{2}} = W_0$,

$$W_0 = U_0 + V_0.$$

By the use of W_0 the seven formulæ may be rendered more regular, but at the cost of introducing another quantity, viz., we have

$$\operatorname{ns} \tfrac{2}{3}K = \tfrac{1}{3}(U_0 + W_0 + N_0 + P_0),$$

$$\operatorname{ds} \tfrac{2}{3}K = \tfrac{1}{3}(-U_0 + V_0 + N_0 + P_0),$$

$$\operatorname{cs} \tfrac{2}{3}K = \tfrac{1}{3}(W_0 + V_0 - N_0 - P_0),$$

$$\operatorname{zs} \tfrac{2}{3}K = \tfrac{1}{3}(N_0 + P_0),$$

$$\operatorname{zc} \tfrac{2}{3}K = \tfrac{1}{3}(-W_0 - V_0 - N_0 - P_0),$$

$$\operatorname{zd} \tfrac{2}{3}K = \tfrac{1}{3}(-U_0 + V_0 - N_0 - P_0),$$

$$\operatorname{zn} \tfrac{2}{3}K = \tfrac{1}{3}(U_0 + W_0 - N_0 - P_0).$$

§ 15. Since $W_0 = U_0 + V_0$ we find from § 10

$$\sqrt{3}\,\{1 + 6\Sigma_1^\infty (-1)^n H(n)\, q^n\} + 8\sqrt{3}\Sigma_1^\infty H(m)\, q^m$$
$$= \sqrt{3}\,\{1 + 6\Sigma_1^\infty H(n)\, q^{4n}\} + 2\sqrt{3}\Sigma_1^\infty H(m)\, q^m = W^{\frac{1}{2}}.$$

and therefore, from § 2,

$$k^2\rho\,\frac{\operatorname{sn}\frac{2}{3}K\operatorname{cn}\frac{2}{3}K}{\operatorname{dn}\frac{2}{3}K} = 4\sqrt{3}\Sigma_1^\infty H(m)\, q^m,$$

$$\rho\,\frac{\operatorname{sn}\frac{2}{3}K\operatorname{dn}\frac{2}{3}K}{\operatorname{cn}\frac{2}{3}K} = \sqrt{3}\,\{1 + 6\Sigma_1^\infty (-1)^n H(n)\, q^n\} + 8\sqrt{3}\Sigma_1^\infty H(m)\, q^m$$
$$= \sqrt{3}\,\{1 + 6\Sigma_1^\infty H(n)\, q^{4n}\} + 2\sqrt{3}\Sigma_1^\infty H(m)\, q^m,$$

$$k'^2\rho\,\frac{\operatorname{sn}\frac{2}{3}K}{\operatorname{cn}\frac{2}{3}K\operatorname{dn}\frac{2}{3}K} = \sqrt{3}\,\{1 + 6\Sigma_1^\infty H(n)\, q^n\} - 8\sqrt{3}\Sigma_1^\infty H(m)\, q^m$$
$$= \sqrt{3}\,\{1 + 6\Sigma_1^\infty H(n)\, q^{4n}\} - 2\sqrt{3}\Sigma_1^\infty H(m)\, q^m.$$

§ 16. By putting $x = \frac{2}{3}K$ in the formulæ

$$\operatorname{zs} x - \operatorname{zn} x = \frac{\operatorname{cn} x \operatorname{dn} x}{\operatorname{sn} x},$$

$$\operatorname{zs} x - \operatorname{zd} x = \frac{\operatorname{cn} x}{\operatorname{sn} x \operatorname{dn} x}.$$

$$\operatorname{zs} x - \operatorname{zc} x = \frac{\operatorname{dn} x}{\operatorname{sn} x \operatorname{cn} x},$$

we deduce from the values of the Zetas in § 13 the following results :

$$\frac{\operatorname{cn}\frac{2}{3}K\operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K} = \tfrac{1}{3}\,(-2U_0 - V_0 + 2N_0 + 2P_0),$$

$$\frac{\operatorname{cn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K\operatorname{dn}\frac{2}{3}K} = \tfrac{1}{3}\,(U_0 - V_0 + 2N_0 + 2P_0).$$

$$\frac{\operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K\operatorname{cn}\frac{2}{3}K} = \tfrac{1}{3}\,(U_0 + 2V_0 + 2N_0 + 2P_0).$$

The right-hand members of these equations may also be written

$$\tfrac{1}{3}\,(-U_0 - W_0 + 2N_0 + 2P_0),$$
$$\tfrac{1}{3}\,(U_0 - V_0 + 2N_0 + 2P_0),$$
$$\tfrac{1}{3}\,(W_0 + V_0 + 2N_0 + 2P_0),$$

respectively.

§ 17. The formulæ used in the preceding section, viz.

$$\operatorname{zs}\tfrac{2}{3}K - \operatorname{zn}\tfrac{2}{3}K = \frac{\operatorname{cn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K}, \text{ \&c.}$$

combined with the equation (§ 12)

$$3 \operatorname{zs}\tfrac{2}{3}K + \operatorname{zc}\tfrac{2}{3}K + \operatorname{zd}\tfrac{2}{3}K + \operatorname{zn}\tfrac{2}{3}K = 0,$$

give the formulæ

$$6 \operatorname{zs}\tfrac{2}{3}K = \frac{\operatorname{cn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K} + \frac{\operatorname{cn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K} + \frac{\operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K \operatorname{cn}\frac{2}{3}K},$$

$$6 \operatorname{zc}\tfrac{2}{3}K = \frac{\operatorname{cn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K} + \frac{\operatorname{cn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K} - 5\,\frac{\operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K \operatorname{cn}\frac{2}{3}K},$$

$$6 \operatorname{zd}\tfrac{2}{3}K = \frac{\operatorname{cn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K} - 5\,\frac{\operatorname{cn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K} + \frac{\operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K \operatorname{cn}\frac{2}{3}K},$$

$$6 \operatorname{zn}\tfrac{2}{3}K = -5\,\frac{\operatorname{cn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K} + \frac{\operatorname{cn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K} + \frac{\operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K \operatorname{cn}\frac{2}{3}K};$$

which express the Zeta functions of $\frac{2}{3}K$ in terms of the elliptic functions of $\frac{2}{3}K$.

It may be remarked also that from the formulæ

$$2 \operatorname{ns} 2x = \frac{\operatorname{cn} x \operatorname{dn} x}{\operatorname{sn} x} + \frac{\operatorname{sn} x \operatorname{dn} x}{\operatorname{cn} x} + k^2\,\frac{\operatorname{sn} x \operatorname{cn} x}{\operatorname{dn} x}, \text{ \&c.}$$

we find, by putting $x = \frac{2}{3}K$,

$$2 \operatorname{ns}\tfrac{2}{3}K = \frac{\operatorname{cn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K} + \frac{\operatorname{sn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K}{\operatorname{cn}\frac{2}{3}K} + k^2\,\frac{\operatorname{sn}\frac{2}{3}K \operatorname{cn}\frac{2}{3}K}{\operatorname{dn}\frac{2}{3}K},$$

$$2 \operatorname{ds}\tfrac{2}{3}K = \frac{\operatorname{cn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K} + \frac{\operatorname{sn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K}{\operatorname{cn}\frac{2}{3}K} - k^2\,\frac{\operatorname{sn}\frac{2}{3}K \operatorname{cn}\frac{2}{3}K}{\operatorname{dn}\frac{2}{3}K},$$

$$2 \operatorname{cs}\tfrac{2}{3}K = -\frac{\operatorname{cn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K} + \frac{\operatorname{sn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K}{\operatorname{cn}\frac{2}{3}K} - k^2\,\frac{\operatorname{sn}\frac{2}{3}K \operatorname{cn}\frac{2}{3}K}{\operatorname{dn}\frac{2}{3}K},$$

§ 18. Since

$$\frac{\operatorname{cn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K} = \frac{k'}{V^{\frac{1}{2}}},$$

$$\frac{\operatorname{cn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K} = \frac{1}{W^{\frac{1}{2}}},$$

$$\frac{\operatorname{dn}\frac{2}{3}K}{\operatorname{sn}\frac{2}{3}K \operatorname{cn}\frac{2}{3}K} = \frac{k}{U^{\frac{1}{2}}},$$

the formulæ of the preceding section give

$$6\,\mathrm{zs}\,\tfrac{2}{3}K = \frac{k}{U^{\frac{1}{2}}} + \frac{1}{W^{\frac{1}{2}}} + \frac{k'}{V^{\frac{1}{2}}},$$

$$6\,\mathrm{zc}\,\tfrac{2}{3}K = -5\,\frac{k}{U^{\frac{1}{2}}} + \frac{1}{W^{\frac{1}{2}}} + \frac{k'}{V^{\frac{1}{2}}},$$

$$6\,\mathrm{zd}\,\tfrac{2}{3}K = \frac{k}{U^{\frac{1}{2}}} - 5\,\frac{1}{W^{\frac{1}{2}}} + \frac{k'}{V^{\frac{1}{2}}},$$

$$6\,\mathrm{zn}\,\tfrac{2}{3}K = \frac{k}{U^{\frac{1}{2}}} + \frac{1}{W^{\frac{1}{2}}} - 5\,\frac{k'}{V^{\frac{1}{2}}};$$

and

$$2\,\mathrm{ns}\,\tfrac{2}{3}K = kU^{\frac{1}{2}} + W^{\frac{1}{2}} + \frac{k'}{V^{\frac{1}{2}}},$$

$$2\,\mathrm{ds}\,\tfrac{2}{3}K = -kU^{\frac{1}{2}} + W^{\frac{1}{2}} + \frac{k'}{V^{\frac{1}{2}}},$$

$$2\,\mathrm{cs}\,\tfrac{2}{3}K = -kU^{\frac{1}{2}} + W^{\frac{1}{2}} - \frac{k'}{V^{\frac{1}{2}}}.$$

§ 19. We may also obtain similar formulæ for $2\,\mathrm{zs}\,\tfrac{2}{3}K$, &c. with $L^{\frac{1}{2}}$, &c., in the denominators; for, it can be shown that

$$3\,\mathrm{zs}\,\tfrac{2}{3}K = \mathrm{ns}\,\tfrac{2}{3}K + \mathrm{ds}\,\tfrac{2}{3}K - \mathrm{cs}\,\tfrac{2}{3}K,$$

$$3\,\mathrm{zc}\,\tfrac{2}{3}K = -2\,\mathrm{ns}\,\tfrac{2}{3}K - 2\,\mathrm{ds}\,\tfrac{2}{3}K - \mathrm{cs}\,\tfrac{2}{3}K,$$

$$3\,\mathrm{zd}\,\tfrac{2}{3}K = -2\,\mathrm{ns}\,\tfrac{2}{3}K + \mathrm{ds}\,\tfrac{2}{3}K + 2\,\mathrm{cs}\,\tfrac{2}{3}K,$$

$$3\,\mathrm{zn}\,\tfrac{2}{3}K = \mathrm{ns}\,\tfrac{2}{3}K - 2\,\mathrm{ds}\,\tfrac{2}{3}K + 2\,\mathrm{cs}\,\tfrac{2}{3}K;$$

and we have

$$\mathrm{ns}\,\tfrac{2}{3}K = \frac{k^{\frac{1}{2}}}{L^{\frac{1}{2}}},\quad \mathrm{ds}\,\tfrac{2}{3}K = \frac{k^{\frac{1}{2}}k'^{\frac{1}{2}}}{M^{\frac{1}{2}}},\quad \mathrm{cs}\,\tfrac{2}{3}K = \frac{k'^{\frac{1}{2}}}{N^{\frac{1}{2}}},$$

whence

$$3\,\mathrm{zs}\,\tfrac{2}{3}K = \frac{k^{\frac{1}{2}}}{L^{\frac{1}{2}}} + \frac{k^{\frac{1}{2}}k'^{\frac{1}{2}}}{M^{\frac{1}{2}}} - \frac{k'^{\frac{1}{2}}}{N^{\frac{1}{2}}},$$

$$3\,\mathrm{zc}\,\tfrac{2}{3}K = -2\,\frac{k^{\frac{1}{2}}}{L^{\frac{1}{2}}} - 2\,\frac{k^{\frac{1}{2}}k'^{\frac{1}{2}}}{M^{\frac{1}{2}}} - \frac{k'^{\frac{1}{2}}}{N^{\frac{1}{2}}},$$

$$3\,\mathrm{zd}\,\tfrac{2}{3}K = -2\,\frac{k^{\frac{1}{2}}}{L^{\frac{1}{2}}} + \frac{k^{\frac{1}{2}}k'^{\frac{1}{2}}}{M^{\frac{1}{2}}} + 2\,\frac{k'^{\frac{1}{2}}}{N^{\frac{1}{2}}},$$

$$3\,\mathrm{zn}\,\tfrac{2}{3}K = \frac{k^{\frac{1}{2}}}{L^{\frac{1}{2}}} - 2\,\frac{k^{\frac{1}{2}}k'^{\frac{1}{2}}}{M^{\frac{1}{2}}} + 2\,\frac{k'^{\frac{1}{2}}}{N^{\frac{1}{2}}}.$$

§ 20. The following formulæ, which were given on p. 162 of the paper in the *Proc. Lond. Math. Soc.* referred to in § 2, may also be noticed here

$$\mathrm{ns}^2 \tfrac{2}{3}K = \tfrac{1}{3}(1 + k^2 + T),$$
$$\mathrm{ds}^2 \tfrac{2}{3}K = \tfrac{1}{3}(k'^2 - k^2 + T),$$
$$\mathrm{cs}^2 \tfrac{2}{3}K = \tfrac{1}{3}(-1 - k'^2 + T),$$

where

$$T = kL - kk'M + k'N.$$

Values of q-series and change of q into q^2, &c., §§ 21—22.

§ 21. To complete the list of values of q-series given in § 10 the two following should be added:

$$\sqrt{3}\{1 + 6\Sigma_1^\infty H(2n)q^{2n}\} = (\tfrac{1}{2}U_0 + V_0)\rho,$$
$$\sqrt{3}\{1 + 6\Sigma_1^\infty H(2n)q^{4n}\} = (N_0 + \tfrac{1}{4}P_0)\rho.$$

These formulæ are easily deducible from those in § 10; for

$$\Sigma_1^\infty H(n)q^n = \Sigma_1^\infty H(2n)q^{2n} + \Sigma_1^\infty H(m)q^m,$$

whence $\quad \Sigma_1^\infty H(2n)q^{2n} = \Sigma_1^\infty H(n)q^n - \Sigma_1^\infty H(m)q^m.$

Also $\quad \Sigma_1^\infty H(2n)q^{4n} = \Sigma_1^\infty H(4n)q^{8n} + \Sigma_1^\infty H(2m)q^{4m}$

$$= \Sigma_1^\infty H(n)q^{8n}.$$

§ 22. It is interesting to show in a tabular form the changes undergone by L_0, U_0, ... (defined in § 10) by the change of q into q^2 or *vice versa*. The following table gives these changes.

q	q^2	q^4	q^8
L_0	$\frac{1}{2}U_0$	$\frac{1}{4}P_0$	
V_0	N_0		
U_0	$\frac{1}{2}P_0$		
R_0	V_0	N_0	
W_0	$N_0 + \frac{1}{2}P_0$		
$2U_0 + V_0$	$N_0 + P_0$		
	$= L_0 + R_0$	$\frac{1}{2}U_0 + V_0$	$N_0 + \frac{1}{4}P_0$

Relations between U, V, W, L, M, N, §§ 23–27.

§ 23. By combining the formulæ we may obtain many relations between U, V, W and L, M, N.

Thus by equating the values of $\dfrac{\text{dn}\,\frac{2}{3}K}{\text{sn}\,\frac{2}{3}K\,\text{cn}\,\frac{2}{3}K}$, &c. in §§ 2 and 16 we find

$$\frac{3k}{U^{\frac{1}{2}}} = U_0 + 2V_0 + 2(N_0 + P_0),$$

$$\frac{3}{W^{\frac{1}{2}}} = U_0 - V_0 + 2(N_0 + P_0),$$

$$\frac{3k'}{V^{\frac{1}{2}}} = -2U_0 - V_0 + 2(N_0 + P_0),$$

which give, by subtraction,

$$\frac{1}{W^{\frac{1}{2}}} - \frac{k'}{V^{\frac{1}{2}}} = kU^{\frac{1}{2}},$$

$$\frac{k}{U^{\frac{1}{2}}} - \frac{1}{W^{\frac{1}{2}}} = k'V^{\frac{1}{2}},$$

$$\frac{k}{U^{\frac{1}{2}}} - \frac{k'}{V^{\frac{1}{2}}} = W^{\frac{1}{2}};$$

or

$$\frac{1}{k}(V^{\frac{1}{2}} - k'W^{\frac{1}{2}}) = \frac{1}{k'}(kW^{\frac{1}{2}} - U^{\frac{1}{2}}) = kV^{\frac{1}{2}} - k'U^{\frac{1}{2}} = (UVW)^{\frac{1}{2}}.$$

§ 24. By multiplying in pairs the formulæ at the end of § 2 we find

$$UW = k^2\,\text{sn}^4\,\tfrac{2}{3}K, \quad UV = k^2k'^2\,\text{sd}^4\,\tfrac{2}{3}K, \quad WV = k'^2\,\text{sc}^4\,\tfrac{2}{3}K,$$

giving the relations

$$UW = L^2, \quad UV = M^2, \quad WV = N^2.$$

From the same formulæ, by division, we find

$$\frac{k'}{k}\,\frac{U^{\frac{1}{2}}}{V^{\frac{1}{2}}} = \text{cn}^2\,\tfrac{2}{3}K = 1 - \text{sn}^2\,\tfrac{2}{3}K, \text{ \&c.},$$

giving

$$\frac{k'}{k}\frac{U^{\frac{1}{2}}}{V^{\frac{1}{2}}}=\tfrac{1}{4}(1-U)^2=1-\frac{L}{k},$$

$$\frac{k}{k'}\frac{V^{\frac{1}{2}}}{U^{\frac{1}{2}}}=\tfrac{1}{4}(1+V)^2=1+\frac{N}{k'},$$

$$\frac{1}{k}\frac{U^{\frac{1}{2}}}{W^{\frac{1}{2}}}=\tfrac{1}{4}(1+U)^2=1-\frac{k'}{k}M,$$

$$k\frac{W^{\frac{1}{2}}}{U^{\frac{1}{2}}}=\tfrac{1}{4}(1+W)^2=1+k'N,$$

$$\frac{1}{k'}\frac{V^{\frac{1}{2}}}{W^{\frac{1}{2}}}=\tfrac{1}{4}(-1+V)^2=1+\frac{k}{k'}M,$$

$$k'\frac{W^{\frac{1}{2}}}{V^{\frac{1}{2}}}=\tfrac{1}{4}(-1+W)^2=1-kL.$$

From these formulæ we may deduce

$$\left(1-\frac{L}{k}\right)\left(1+\frac{N}{k'}\right)=1,$$

$$\left(1-\frac{k'}{k}M\right)(1+k'N)=1,$$

$$\left(1+\frac{k}{k'}M\right)(1-kL)=1,$$

giving

$$LN=kN-k'L,$$

$$k'MN=kN-M,$$

$$kLM=M-k'L;$$

any one of which may be deduced from the other two.

We may also derive from these equations the relation

$$\frac{k'}{L}+\frac{k}{W}=\frac{1}{M},$$

which may be easily verified, for it is equivalent to

$$\frac{k'}{k\,\mathrm{sn}^2\frac{2}{3}K}+\frac{k}{k'\,\mathrm{sc}^2\frac{2}{3}K}=\frac{1}{kk'\,\mathrm{sd}^2\frac{2}{3}K}.$$

§ 25. In connection with this last group of formulæ we may notice also the somewhat similar relations involving

U, V, W, which may be derived from § 2, viz.,

$$UV + U - V = -3,$$
$$VW - V - W = \quad 3,$$
$$UW + U + W = \quad 3,$$

from which we may deduce

$$UW + VW - UV = 9.$$

Another formula involving U and V may also be noticed, viz.,

$$k^2(1+U)^2 + k'^2(-1+V)^2 = 4,$$

which may be derived from the formulæ in the preceding section combined with

$$kU^{\frac{1}{2}} + k'V^{\frac{1}{2}} = W^{\frac{1}{2}} \quad (\S\ 14).$$

§ 26. From the general formulæ,

$$\operatorname{sn} 2x = \frac{2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x}{1 - k^2 \operatorname{sn}^4 x},$$

$$\operatorname{sd} 2x = \frac{2 \operatorname{sd} x \operatorname{cd} x \operatorname{nd} x}{1 + k^2 k'^2 \operatorname{sd}^4 x},$$

$$\operatorname{sc} 2x = \frac{2 \operatorname{sc} x \operatorname{nc} x \operatorname{dc} x}{1 - k'^2 \operatorname{sc}^4 x},$$

by putting $x = \frac{2}{3}K$ and using

$$\operatorname{sn} \tfrac{4}{3}K = \operatorname{sn} \tfrac{2}{3}K, \quad \operatorname{sd} \tfrac{4}{3}K = \operatorname{sd} \tfrac{2}{3}K, \quad \operatorname{sc} \tfrac{4}{3}K = -\operatorname{sc} \tfrac{2}{3}K,$$

we find
$$1 - k^2 \operatorname{sn}^4 \tfrac{2}{3}K = 2 \operatorname{cn} \tfrac{2}{3}K \operatorname{dn} \tfrac{2}{3}K,$$
$$1 + k^2 k'^2 \operatorname{sd}^4 \tfrac{2}{3}K = 2 \operatorname{nd} \tfrac{2}{3}K \operatorname{cd} \tfrac{2}{3}K,$$
$$1 - k'^2 \operatorname{sc}^4 \tfrac{2}{3}K = -2 \operatorname{dc} \tfrac{2}{3}K \operatorname{nc} \tfrac{2}{3}K.$$

§ 27. Expressing these formulæ in terms of L, M, N, U, V, W, they become

$$1 - L^2 = \tfrac{1}{2}(-1 + U + W - UW),$$
$$1 + M^2 = \tfrac{1}{2}(-1 - U + V + UV),$$
$$-1 + N^2 = \tfrac{1}{2}(1 + V + W + VW);$$

and combining these equations with

$$U + W + UW = \quad 3,$$
$$U - V + UV = -3,$$
$$V + W - VW = -3,$$

which were obtained in § 22, we find

$$U+W=\ \ 3-L^2,$$

$$U-V=-3-M^2,$$

$$V+W=-3+N^2,$$

besides $UW=L^2$, $UV=M^2$, $VW=N^2$, which have been already obtained in § 22.

From these equations we see that

$$L^2-M^2+N^2=9,$$

and we may also deduce the formulæ

$$U^{\frac{1}{2}}+W^{\frac{1}{2}}=\surd\{(L+1)(3-L)\},$$

$$V^{\frac{1}{2}}+W^{\frac{1}{2}}=\surd\{(N-1)(N+3)\}.$$

Values of $k\rho^2 \operatorname{cn}\frac{2}{3}K \operatorname{dn}\frac{2}{3}K$, &c.. and of $k^8\rho^8 \operatorname{sn}^4\frac{2}{3}K$, &c., as q-series and in terms of k, §§ 28–38.

§ 28. Starting with the formulæ*

$$k\rho \operatorname{sn}\rho x=4\Sigma_1^\infty.\Sigma_m \sin\delta x.q^{\frac{1}{2}m},$$

$$kk'\rho \operatorname{sd}\rho x=4\Sigma_1^\infty(-1)^{\frac{1}{2}(m-1)}.\Sigma_m \sin\delta x.q^{\frac{1}{2}m},$$

$$k'\rho \operatorname{sc}\rho x=\tan x-4\Sigma_1^\infty.\Sigma_n(-1)^{d+d'}\sin 2dx.q^{2n},$$

where $\Sigma_m \sin\delta x$ denotes $\sin\delta_1 x+\sin\delta_2 x+\ldots$, δ_1, δ_2, ... being all the divisors of m (which are all uneven), and $\Sigma_n(-1)^{d+d'}\sin 2dx$ denotes $(-1)^{d_1+d_1'}\sin 2d_1x+(-1)^{d_2+d_2'}\sin 2d_2x+\ldots$, d_1, d_2, ... being all the divisors of n, and d_1', d_2', ... being their conjugates; differentiating with respect to x, and putting $x=\frac{2}{3}K$, we find

$$k\rho^2 \operatorname{cn}\tfrac{2}{3}K \operatorname{dn}\tfrac{2}{3}K=4\Sigma_1^\infty.\Sigma_m\delta\cos\tfrac{1}{3}\delta\pi.q^{\frac{1}{2}m},$$

$$kk'\rho^2 \operatorname{nd}\tfrac{2}{3}K \operatorname{cd}\tfrac{2}{3}K=4\Sigma_1^\infty(-1)^{\frac{1}{2}(m-1)}.\Sigma_m\delta\cos\tfrac{1}{3}\delta\pi.q^{\frac{1}{2}m},$$

$$k'\rho^2 \operatorname{dc}\tfrac{2}{3}K \operatorname{nc}\tfrac{2}{3}K=\sec^2x-8\Sigma_1^\infty.\Sigma_n(-1)^{d+d'}d\cos\tfrac{2}{3}d\pi.q^{2n}.$$

§ 29. To calculate the value of $\Sigma_m\delta\cos\frac{1}{3}\delta\pi$ we first notice that, since δ is uneven,

$$\cos\tfrac{1}{3}\delta\pi=\tfrac{1}{2},\text{ if }\delta\text{ is not divisible by 3,}$$

and $$=-1=\tfrac{1}{2}-\tfrac{3}{2},\text{ if }\delta\text{ is divisible by 3.}$$

Thus $$\Sigma_m\delta\cos\tfrac{1}{3}\delta\pi=\tfrac{1}{2}\Sigma_m\delta-\tfrac{3}{2}\Sigma_m\epsilon,$$

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* *Messenger*, Vol. XVIII., p. 9.

where ϵ is any divisor of m which is divisible by 3 (so that the second term occurs only when m is divisible by 3).

When m is divisible by 3, let it $=3\mu$; then $\epsilon_r=3\eta_r$, where η_r is any divisor of μ. Then the equation becomes

$$\begin{aligned}\Sigma_m\delta\cos\tfrac{1}{3}\delta\pi &= \tfrac{1}{2}\Sigma_m\delta-\tfrac{9}{2}\Sigma_\mu\eta\\ &= \tfrac{1}{2}\Delta(m)-\tfrac{9}{2}\Delta(\mu)\\ &= \tfrac{1}{2}\Delta(m)-\tfrac{9}{2}\Delta(\tfrac{1}{3}m).\end{aligned}$$

where $\Delta(n)$ denotes the sum of the uneven divisors of n.

This equation also holds good when m is not divisible by 3 if we define $\Delta(n)$ as zero when n is not an integer.

§ 30. To calculate $\Sigma_n(-1)^{d+d'}d\cos\tfrac{2}{3}d\pi$, we first notice that

$$\cos\tfrac{2}{3}d\pi=-\tfrac{1}{2},\text{ if } d \text{ is not divisible by 3,}$$

and

$$=-\tfrac{1}{2}+\tfrac{3}{2},\text{ if } d \text{ is divisible by 3,}$$

so that

$$\Sigma_n(-1)^{d+d'}d\cos\tfrac{2}{3}d\pi=-\tfrac{1}{2}\Sigma_n(-1)^{d+d'}d+\tfrac{3}{2}\Sigma_n(-1)^{e+e'}e,$$

where e is any divisor of n which is divisible by 3, and e' is its conjugate.

Now if $\delta_1, \delta_2, \ldots$ are the uneven divisors of n, and if $n=2^i m$, m being uneven and $i>0$, then all the divisors of n are

$$\delta_1,\ \delta_2,\ \ldots,\ 2\delta_1,\ 2\delta_2,\ \ldots,\ 2^2\delta_1,\ 2^2\delta_2,\ \ldots,\ \ldots,\ 2^i\delta_1,\ 2^i\delta_2,\ \ldots.$$

Therefore

$$\begin{aligned}\Sigma_n(-1)^{d+d'}d &= -\Delta(n)+2\Delta(n)+2^2\Delta(n)+\ldots+2^{i-1}\Delta(n)-2^i\Delta(n)\\ &= (1+2+\ldots+2^i)\,\Delta(n)-2\,(1+2^i)\,\Delta(n)\\ &= (2^{i+1}-1-2-2^{i+1})\,\Delta(n)=-3\Delta(n).\end{aligned}$$

§ 31. Now suppose that n is divisible by 3 and let $n=3\nu$. Let $e_r=3\epsilon_r$; then ϵ_r is any divisor of ν, and

$$e_r'=\frac{n}{e_r}=\frac{3\nu}{3\epsilon_r}=\frac{\nu}{\epsilon_r}=\epsilon_r',$$

where ϵ_r' is the conjugate of ϵ_r with respect to ν. Thus

$$\Sigma_n(-1)^{e+e'}e=3\Sigma_\nu(-1)^{\epsilon+\epsilon'}\epsilon,$$

whence, by the preceding section, if $n=2^i m$ $(i>0)$,

$$\Sigma_n(-1)^{e+e'}=-3.3\Delta(\nu)=-9\Delta(\tfrac{1}{3}n).$$

Therefore, if $n=2^i m$, $(i>0)$,

$$\Sigma_n (-1)^{d+d'} d \cos \tfrac{2}{3} d\pi = \tfrac{3}{2}\Delta(n) - \tfrac{27}{2}\Delta(\tfrac{1}{3}n).$$

This formula is still true when n is not divisible by 3 as $\Delta(\frac{1}{3}n)$ is then zero (§ 29).

§ 32. If $i=0$, that is, if n is uneven,

$$\begin{aligned}\Sigma_n (-1)^{d+d'} d \cos \tfrac{2}{3} d\pi &= -\tfrac{1}{2}\Sigma_n \delta + \tfrac{3}{2}\Sigma_n e \\ &= -\tfrac{1}{2}\Delta(n) + \tfrac{9}{2}\Delta(\tfrac{1}{3}n),\end{aligned}$$

which is also true when n is not divisible by 3.

Combining this result with that obtained in the preceding section, we find

$$\Sigma_n (-1)^{d+d'} d \cos \tfrac{2}{3} d\pi = \tfrac{1}{2}\{1+(-1)^n 2\}\{\Delta(n) - 9\Delta(\tfrac{1}{3}n)\}.$$

§ 33. The three q-expansions therefore are

$$k\rho^2 \operatorname{cn} \tfrac{2}{3}K \operatorname{dn} \tfrac{2}{3}K = 2\Sigma_1^\infty \{\Delta(m) - 9\Delta(\tfrac{1}{3}m)\} q^{\frac{1}{2}m},$$

$$kk'\rho^2 \operatorname{nd} \tfrac{2}{3}K \operatorname{cd} \tfrac{2}{3}K = 2\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} \{\Delta(m) - 9\Delta(\tfrac{1}{3}m)\} q^{\frac{1}{2}m},$$

$$k'\rho^2 \operatorname{dc} \tfrac{2}{3}K \operatorname{nc} \tfrac{2}{3}K = 4 - 4\Sigma_1^\infty \{1+(-1)^n 2\}\{\Delta(n) - 9\Delta(\tfrac{1}{3}n)\} q^{2n}.$$

§ 34. By virtue of the formulæ at the end of § 26, these equations show also that

$$k\rho^2 (1 - k^2 \operatorname{sn}^4 \tfrac{2}{3}K) = 4\Sigma_1^\infty \{\Delta(m) - 9\Delta(\tfrac{1}{3}m)\} q^{\frac{1}{2}m},$$

$$kk'\rho^2 (1 + k^2 k'^2 \operatorname{sd}^4 \tfrac{2}{3}K) = 4\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} \{\Delta(m) - 9\Delta(\tfrac{1}{3}m)\} q^{\frac{1}{2}m},$$

$$k'\rho^2 (-1 + k'^2 \operatorname{sc}^4 \tfrac{2}{3}K) = 8 - 8\Sigma_1^\infty \{1+(-1)^n 2\}\{\Delta(n) - 9\Delta(\tfrac{1}{3}n)\} q^{2n}.$$

Now

$$k\rho^2 = 4\Sigma_1^\infty \Delta(m) q^{\frac{1}{2}m},$$

$$kk'\rho^2 = 4\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} \Delta(m) q^{\frac{1}{2}m},$$

$$k'\rho^2 = 1 + 8\Sigma_1^\infty \{1+(-1)^n 2\} \Delta(n) q^{2n},*$$

whence, by subtraction or addition, we find

$$\begin{aligned} k^3\rho^2 \operatorname{sn}^4 \tfrac{2}{3}K &= \quad 36\Sigma_1^\infty \Delta(\tfrac{1}{3}m) q^{\frac{1}{2}m}, \\ k^3k'^3\rho^2 \operatorname{sd}^4 \tfrac{2}{3}K &= -36\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} \Delta(\tfrac{1}{3}m) q^{\frac{1}{2}m}, \\ k'^3\rho^2 \operatorname{sc}^4 \tfrac{2}{3}K &= \quad 9 + 72\Sigma_1^\infty \{1+(-1)^n 2\} \Delta(\tfrac{1}{3}n) q^{2n}; \end{aligned}$$

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* These formulæ are given on pp. 162–163 of Vol. XX. of the *Quarterly Journal* in a paper "On the function $\chi(n)$."

which may be more conveniently written

$$k^2\rho^2 \operatorname{sn}^4 \tfrac{2}{3}K = 36\,\Sigma_1^\infty \Delta(\tfrac{1}{3}m)\, q^{\frac{1}{2}m},$$

$$k^3k'^3\rho^2 \operatorname{sd}^4 \tfrac{2}{3}K = 36\,\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)}\, \Delta(m)\, q^{\frac{3}{2}m},$$

$$k'^3\rho^2 \operatorname{sc}^4 \tfrac{2}{3}K = 9 + 72\Sigma_1^\infty \{1 + (-1)^n 2\}\, \Delta(n)\, q^{6n}.$$

§ 35. Since

$$k \operatorname{sn}^2 \tfrac{2}{3}K = L, \quad kk' \operatorname{sd}^2 \tfrac{2}{3}K = M, \quad k' \operatorname{sc}^2 \tfrac{2}{3}K = N,$$

the values of these q-series are

$$kL^2\rho^2, \quad kk'M^2\rho^2, \quad k'N^2\rho^2$$

respectively, and the values of the q-series in § 33 are

$$\tfrac{1}{2}(1 - L^2), \quad \tfrac{1}{2}(1 + M^2), \quad \tfrac{1}{2}(-1 + N^2)$$

respectively.

The values of these latter quantities in terms of λ, μ, ν (§ 1) are

$$\tfrac{1}{2}(1-L^2) = \sqrt{\{2(1+\lambda+\lambda^3+\lambda^4)^{\frac{1}{2}}+2+\lambda-\lambda^2)\}} - \sqrt{(1-\lambda+\lambda^2)} - 1,$$

$$\tfrac{1}{2}(1+M^2) = \mp\sqrt{\{2(1-\mu-\mu^3+\mu^4)^{\frac{1}{2}}+2-\mu-\mu^2)\}} + \sqrt{(1+\mu+\mu^2)} - 1,$$

$$\tfrac{1}{2}(-1+N^2) = \sqrt{\{2(1+\nu+\nu^3+\nu^4)^{\frac{1}{2}}+2+\nu-\nu^2\}} + \sqrt{(1-\nu+\nu^2)} + 1,$$

where in the middle formula the upper or lower sign is to be taken according as $k <$ or $> k'$.

§ 36. The expansions given in § 6, viz.

$$k^2\rho\, \frac{\operatorname{sn}^2 \frac{2}{3}K \operatorname{cn}^2 \frac{2}{3}K}{\operatorname{dn}^2 \frac{2}{3}K} = 12\Sigma_1^\infty E(m)\, q^{\frac{3}{2}m},$$

$$\rho\, \frac{\operatorname{sn}^2 \frac{2}{3}K \operatorname{dn}^2 \frac{2}{3}K}{\operatorname{cn}^2 \frac{2}{3}K} = 3\,\{1 + 4\Sigma_1^\infty E(n)\, q^{3n}\},$$

$$k'^2\rho\, \frac{\operatorname{sn}^2 \frac{2}{3}K}{\operatorname{cn}^2 \frac{2}{3}K \operatorname{dn}^2 \frac{2}{3}K} = 3\,\{1 + 4\Sigma_1^\infty (-1)^n E(n)\, q^{3n}\},$$

might have been derived at once from

$$k\rho = 4\Sigma_1^\infty E(m)\, q^{\frac{1}{2}m},$$

$$\rho = 1 + 4\Sigma_1^\infty E(n)\, q^n,$$

$$k'\rho = 1 + 4\Sigma_1^\infty (-1)^n E(n)\, q^n,$$

by changing q into q^3; and by the same change the expansions obtained in § 34 might have been derived from

$$k\rho^2 = 4\Sigma_1^\infty \Delta(m)\, q^{\frac{1}{2}m},$$

$$kk'\rho^2 = 4\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)}\Delta(m)\, q^{\frac{1}{2}m},$$

$$k'\rho^2 = 1 + 8\Sigma_1^\infty \{1 + (-1)^n 2\}\, \Delta(m)\, q^{2n},$$

respectively.

§ 37. The formulæ required to make these changes are deducible from Jacobi's general formulæ of transformation by putting $n = 3$.

Denoting by k_3, k_3' ρ_3 the quantities into which k, k', ρ are converted by the change of q into q^3 we have*

$$k_3 = k^3 \operatorname{cd}^4 \tfrac{2}{3}K, \quad k_3' = k'^3 \operatorname{nd}^4 \tfrac{2}{3}K,$$

$$\rho_3 = \tfrac{1}{3}\rho\, \frac{\operatorname{sd}^2 \frac{2}{3}K}{\operatorname{cd}^2 \frac{2}{3}K \operatorname{nd}^2 \frac{2}{3}K},$$

whence

$$k_3\rho_3 = \tfrac{1}{3}k^3\rho\, \frac{\operatorname{sd}^2 \frac{2}{3}K \operatorname{cd}^2 \frac{2}{3}K}{\operatorname{nd}^2 \frac{2}{3}K} = \tfrac{1}{3}k^3\rho\, \frac{\operatorname{sn}^2 \frac{2}{3}K \operatorname{cn}^2 \frac{2}{3}K}{\operatorname{dn}^2 \frac{2}{3}K},$$

$$\rho_3 = \tfrac{1}{3}\rho\, \frac{\operatorname{sd}^2 \frac{2}{3}K}{\operatorname{cd}^2 \frac{2}{3}K \operatorname{nd}^2 \frac{2}{3}K} = \tfrac{1}{3}\rho\, \frac{\operatorname{sn}^2 \frac{2}{3}K \operatorname{dn}^2 \frac{2}{3}K}{\operatorname{cn}^2 \frac{2}{3}K},$$

$$k_3'\rho_3 = \tfrac{1}{3}k'^3\rho\, \frac{\operatorname{sd}^2 \frac{2}{3}K \operatorname{nd}^2 \frac{2}{3}K}{\operatorname{cd}^2 \frac{2}{3}K} = \tfrac{1}{3}k'^3\rho\, \frac{\operatorname{sn}^2 \frac{2}{3}K}{\operatorname{cn}^2 \frac{2}{3}K \operatorname{dn}^2 \frac{2}{3}K};$$

and

$$k_3\rho_3^2 = \tfrac{1}{9}k^3\rho^2\, \frac{\operatorname{sd}^4 \frac{2}{3}K}{\operatorname{nd}^4 \frac{2}{3}K} = \tfrac{1}{9}k^3\rho^2 \operatorname{sn}^4 \tfrac{2}{3}K,$$

$$k_3k_3'\rho_3^2 = \tfrac{1}{9}k^3k'^3\rho^2 \operatorname{sd}^4 \tfrac{2}{3}K = \tfrac{1}{9}k^3k'^3\rho^2 \operatorname{sd}^4 \tfrac{2}{3}K,$$

$$k_3'\rho_3^2 = \tfrac{1}{9}k'^3\rho^2\, \frac{\operatorname{sd}^4 \frac{2}{3}K}{\operatorname{cd}^4 \frac{2}{3}K} = \tfrac{1}{9}k'^3\rho^2 \operatorname{sc}^4 \tfrac{2}{3}K.$$

§ 38. In the same manner, taking for example the list of q-series and their values on pp. 162–163 of Vol. XX. of the *Quarterly Journal*, we may deduce the value of any of these q-series when q is replaced by q^3. For example, from

$$k^{\frac{3}{2}}\rho^3 = -\Sigma_3^\infty E_2(r)\, q^{\frac{1}{4}r},$$

where r is any number of the form $4k + 3$, we deduce

$$k^{\frac{9}{2}}\rho^3 \operatorname{sn}^6 \tfrac{2}{3}K = -27\,\Sigma_3^\infty E_2(r)\, q^{\frac{3}{4}r}.$$

* *Proc. Lond. Math. Soc.*, Vol. XXII., p. 151.

ON FINITE GROUPS IN WHICH ALL THE SYLOW SUBGROUPS ARE CYCLICAL.

By *W. Burnside.*

1. I HAVE shown in my *Theory of Groups* (p. 352) that a group of order $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$, where $p_1, p_2, \ldots, p_n$ are primes in ascending order, for which the subgroups of order $p_r^{\alpha_r}$ $(r = 1, 2, \ldots, n)$ are cyclical, contains characteristic subgroups of orders $p_s^{\alpha_s} p_{s+1}^{\alpha_{s+1}} \ldots p_n^{\alpha_n}$, $(s = 2, 3, \ldots, n)$; and that all such groups are soluble.

Let
$$P_1,\ P_2,\ \ldots,\ P_n,$$
be operations of orders
$$p_1^{\alpha_1},\ p_2^{\alpha_2},\ \ldots,\ p_n^{\alpha_n},$$
belonging to such a group G. The characteristic subgroup $\{P_{n-1}, P_n\}$ contains the cyclical subgroup $\{P_n\}$ self-conjugately. Suppose that $p_{n-1}^{\alpha_{n-1}-a_{n-1}}$th power of P_{n-1} is the lowest power that is permutable with P_n. Then $\{P_{n-1}, P_n\}$ contains a cyclical subgroup $\{P'\}$ of order $p_{n-1}^{a_{n-1}} p_n^{\alpha_n}$ as a characteristic subgroup; and no power of P_{n-1} lower than the $p_{n-1}^{\alpha_{n-1}-a_{n-1}}$th is permutable with P'. The cyclical subgroup $\{P'\}$ is a characteristic subgroup of G. Suppose now that the $p_{n-2}^{\alpha_{n-2}-a_{n-2}}$th power of P_{n-2} is the lowest that is permutable with P'. Then the subgroup $\{P_{n-2}, P_{n-1}, P_n\}$ contains a cyclical subgroup $\{P''\}$ of order $p_{n-2}^{a_{n-2}} p_{n-1}^{a_{n-1}} p_n^{\alpha_n}$ as a characteristic subgroup. In fact $p_{n-2}^{a_{n-2}} p_{n-1}^{a_{n-1}} p_n^{\alpha_n}$ is the order of the greatest subgroup of $\{P_{n-2}, P_{n-1}, P_n\}$ which contains P' self-conjugately, and the subgroup being Abelian is necessarily cyclical. Moreover, no operation of $\{P_{n-2}, P_{n-1}, P_n\}$ which is not contained in $\{P''\}$ can be permutable with P''. Suppose in fact that
$$P_{n-1}^{-1} P'' P_{n-1} = P''^{x},$$
so that, if $\alpha_{n-1} - a_{n-1} = b$, $x^{p_{n-1}^{b}}$ is the first power of x which is congruent to unity mod. m $(m = p_{n-2}^{a_{n-2}} p_{n-1}^{a_{n-1}} p_n^{\alpha_n})$; and that
$$P_{n-2}^{-1} P'' P_{n-2} = P''^{y},$$
$y^{p_{n-2}^{c}}$ $(c = \alpha_{n-2} - a_{n-2})$ being the first power of y congruent to unity mod. m.

Then $$P_{n-2}^{-j}P_{n-1}^{-i}P''P_{n-1}^{i}P_{n-2}^{j}=P''^{x^iy^j},$$

and if $P_{n-1}^{i}P_{n-2}^{j}$ is permutable with P'',

$$x^iy^j=1 \pmod{m}.$$

This involves, however, that i must be a multiple of p_{n-1}^{b} and j of p_{n-2}^{c}; in which case $P_{n-1}^{i}P_{n-2}^{j}$ is contained in $\{P''\}$. The process by which P' and P'' have been formed may be continued till we arrive at an operation Q of order

$$\mu=p_1^{a_1}p_2^{a_2}, \ldots, p_n^{a_n};\ a_1\leq\alpha_1;\ a_2\leq\alpha_2, \ldots, a_{n-1}\leq\alpha_{n-1},$$

such that the cyclical group $\{Q\}$ is a characteristic subgroup of G, while no operation of G which is not contained in $\{Q\}$ is permutable with Q. Put now

$$b_r=\alpha_r-a_r.$$

Then if b_r is not zero P_r is not permutable with Q, and

$$P_r^{-1}QP_r=Q^{x_r},$$

where $x_r^{p_r^{b_r}}$ is the first power of x_r which is congruent to unity mod. μ; and

$$P_{n-1}^{-1}P_{n-2}^{-1}\ldots P_1^{-1}QP_1P_2\ldots P_{n-1}=Q^{x_1, x_2, \ldots, x_{n-1}}.$$

Now if $$(x_1x_2\ldots x_{n-1})^i\equiv 1, \text{ mod. } \mu,$$

then $$x_1^i\equiv 1,\ x_2^i\equiv 1,\ \ldots,\ x_{n-1}^i\equiv 1, \text{ mod. } \mu,$$

and i must be equal to or a multiple of

$$\nu=p_1^{b_1}p_2^{b_2}\ldots p_{n-1}^{b_{n-1}}.$$

The lowest power* of R, or

$$P_1P_2\ldots P_{n-1},$$

which is permutable with Q is the ν^{th}, and R^ν, being permutable with Q, must be contained in $\{Q\}$. Now μ, the order of Q, and ν are such that $\mu\nu$ is the order of G. Hence G may be defined by relations of the form

$$Q^\mu=1,\ R^{-1}QR=Q^s,\ R^\nu=Q^t,$$

where μ is the product of $p_n^{a_n}$ and a factor of

$$p_1^{a_1}p_2^{a_2}\ldots p_{n-1}^{a_{n-1}},$$

* When some of the b's are zero, the corresponding p's do not occur in ν, and the corresponding P's are to be omitted from R.

while ν is the complementary factor of the latter number. Further s is such that s^ν is the lowest power of s which is congruent to unity mod. μ; and t is a suitably determined integer, whose form is further considered in the next paragraph.

2. I proceed next to consider the representations of a group, all of whose Sylow subgroups are cyclical, as an irreducible group of linear substitutions. The notation of § 1 will be adhered to. Since Q is an operation of order μ there must be some irreducible representation in which Q corresponds to a linear substitution whose order is actually μ and not a factor of μ. Suppose, if possible, that this representation is not simply-isomorphic with G. Then G must have a self-conjugate subgroup H such that the representation is simply-isomorphic with G/H. But if both $\{Q\}$ and H are self-conjugate subgroups of G there would be operations of G, not belonging to $\{Q\}$ and permutable with Q, which is not the case. Hence an irreducible representation of G, in which Q is given by a linear substitution of order μ, is simply-isomorphic with G. In such a representation let Q be given by

$$x_1' = \omega_1 x_1, \quad x_2' = \omega_2 x_2, \ldots, x_k' = \omega_k x_k.$$

The ω's are μth roots of unity, of which at least one is a primitive μth root. Suppose, if possible, that $\omega_1, \omega_2, \ldots, \omega_i$ were primitive μ'th roots, μ' being a factor of μ, the remainder being primitive μth roots. Then $Q^{\mu'}$ leaves $x_1, x_2, \ldots, x_i$ unchanged; and, since $\{Q^{\mu'}\}$ is a self-conjugate subgroup of G, every operation of G transforms $x_1, x_2, \ldots, x_i$ among themselves. The group being irreducible this is impossible, and hence all the ω's are primitive μth roots. Suppose next that $\omega_1, \omega_2, \ldots, \omega_i$ were all equal; since $\{Q\}$ is a self-conjugate subgroup, the remaining ω's must be equal in sets of i. If

$$\omega_1 = \omega_2 = \ldots = \omega_i = \omega.$$

then because $R^{-1}QR = Q^s$, there must be a second set of x's, say

$$x_{i+1}, x_{i+2}, \ldots, x_{2i}$$

for which the multipliers are all ω^s, and R must change $x_1, x_2, \ldots, x_i$ into linear functions of $x_{i+1}, x_{i+2}, \ldots, x_{2i}$. Without altering the form of Q we may write for R, so far as it affects $x_1, x_2, \ldots, x_i$,

$$x_1' = x_{i+1}, \quad x_2' = x_{i+2}, \ldots, x_i' = x_{2i}.$$

Similarly, there must be another set of i x's, for each of which the multiplier is ω^{s^2}, and R must replace $x_{i+1}, x_{i+2}, \ldots, x_{2i}$ by linear functions of the next set. Since R^ν is the first power of R which occurs in $\{Q\}$, there must be ν such sets of x's; and except for the last set the form of R may be taken as above. Hence R is

$$x'_{ki+1} = x_{(k+1)i+1},\ x'_{ki+2} = x_{(k+1)i+2},\ \ldots,\ x'_{(k+1)i} = x_{(k+2)i}$$

$$(k = 0, 1, 2, \ldots, \nu - 2)$$

$$x'_{(\nu-1)i+u} = \sum_{v=1}^{v=i} a_{uv} x_v$$

$$(u = 1, 2, \ldots, i).$$

But, since R^ν belongs to $\{Q\}$,

$$x_u' = \sum_{v=1}^{v=i} a_{uv} x_v$$

$$(u = 1, 2, \ldots, i)$$

must replace each x by the same multiple of itself. Hence

$$a_{uv} = 0\,(u \neq v),\ a_{uu} = \omega'.$$

If then $i > 1$, the symbols $x_1, x_{i+1}, \ldots, x_{(\nu-1)i+1}$ are transformed among themselves by both Q and R, and the group is reducible. Hence $i = 1$, and Q, R are given (writing $x_2, x_3, \ldots$, for $x_{i+1}, x_{2i+1}, \ldots$) by

$$Q \sim x_1' = \omega x_1,\ x_2' = \omega^s x_2,\ \ldots,\ x_\nu' = \omega^{s^{\nu-1}} x_\nu,$$

$$R \sim x_1' = x_2,\quad x_2' = \quad x_3,\ \ldots,\ x_\nu' = \omega' x_1.$$

Finally, the relation

$$R^\nu = Q^t$$

involves $$\omega' = \omega^t = \omega^{st} = \ldots,$$

and therefore $(s-1)t \equiv 0$, mod. μ, while s is subject to the previously given condition $s^\nu \equiv 1$, mod. μ.

The multipliers of R are the ν distinct νth roots of ω', and those of R^i are the ith powers of these. Hence the necessary and sufficient condition that no power of R shall have a unit multiplier is that, if p_r is a factor of ν, it is also a factor of the index to which the root ω' belongs. This is equivalent to the condition that the order of Q is divisible by each of the primes which occur in the order of G. This then is the necessary and sufficient condition that, in the irreducible group of linear substitutions, no substitution except the identical one shall have a unit multiplier. For every

substitution is conjugate either to a power of R or a power of Q, and it is obvious that no power of Q other than identity has a unit multiplier.

3. Closely analogous, but necessarily less simple, results hold when, the order of G being even, the subgroups of order 2^a instead of being cyclical are of the other type which contains only one operation of order 2, viz.,

$$S^{2^{a-1}}=1, \quad T^2=S^{2^{a-2}}, \quad T^{-1}ST=S^{-1}.$$

When $a>3$ this group admits of no isomorphism of odd order; and the reasoning used in the former case* still holds to shew that the group has characteristic subgroups of orders $p_s^{a_s}p_{s+1}^{a_{s+1}}\dots p_n^{a_n}$ $(s=2, 3, \dots, n)$. The characteristic subgroup of odd order $p_2^{a_2}p_3^{a_3}\dots p_n^{a_n}$ may then be regarded as defined by the relations at the end of § 1. Now the group of isomorphisms of $\{Q\}$, a cyclical group of odd order, is necessarily Abelian. Hence Q must be permutable with every operation of some subgroup g of $\{S, T\}$, in respect of which the factor-group $\{S, T\}/g$ is Abelian. There are four subgroups of $\{S, T\}$ satisfying this condition, viz., (i) $\{S, T\}$ itself, (ii) $\{S\}$, (iii) $\{S^2, T\}$, (iv) $\{S^2\}$; and in respect of each of these there may be a distinct type of group. A complete discussion of the various cases that may arise, depending as they do on the form of the subgroup $\{Q, R\}$, would be extremely long, and it is not proposed to enter on it. When these groups are represented as irreducible groups of linear substitutions, it may again happen that no substitution except the identical one has a unit multiplier. That this is so may be seen from the simplest cases, *e.g.*, the group generated by

$$x'=\alpha x, \qquad y'=\alpha^{-1}y;$$

and

$$x'=y, \qquad y'=-x;$$

where α is a primitive $2^n(2m+1)$th root of unity.

When in the relations defining $\{S, T\}$ a is 3, the group is the quaternion group, and it admits an isomorphism of order 3. In this case it is not necessarily true that the group has characteristic subgroups of the orders

$$p_s^{a_s}p_{s+1}^{a_{s+1}}\dots p_n^{a_n} (s=2, 3, \dots, n),$$

nor is it necessarily soluble. The group of order 120 with a self-conjugate subgroup of order 2, in respect of which it is simply isomorphic with the icosahedral group, is a case in point.

* *Theory of Groups, loc. cit.*

ON A GENERAL PROPERTY OF FINITE IRREDUCIBLE GROUPS OF LINEAR SUBSTITUTIONS.

By *W. Burnside.*

1. In general a finite irreducible group of linear substitutions has the property that for some subgroup, other than that consisting of the identical substitution only, there is a linear function of the variables operated on which remains absolutely invariant. I propose here to determine and classify those irreducible groups which do not possess this property. I first deal with the case in which the order of the group is the power of a prime and then proceed to the general case.

2. I denote by Γ a finite irreducible group of linear substitutions, and by G the abstract group which is simply isomorphic with Γ. The substitution of Γ, which corresponds to an operation S of order m belonging to G, may be reduced to the canonical form

$$x_1' = \omega_1 x_1,\ x_2' = \omega_2 x_2,\ \ldots,\ x_n' = \omega_n x_n,$$

where n is the number of variables and the ω's are mth roots of unity.

If Γ has not the property stated, as is now assumed, then each ω must be a primitive mth root of unity. For if ω_1 were a m/pth root of unity (p a factor of m), then x_1 would be an absolute invariant for $S^{m/p}$, and therefore for the subgroup generated by $S^{m/p}$.

If S belong to the ith conjugate set in G, and χ_i is the corresponding characteristic in Γ, an irreducible representation of G, then

$$\chi_i = \omega_1 + \omega_2 + \ldots + \omega_n,$$

a rational function of an assigned primitive mth root ω. When ω is replaced by the other primitive mth roots of unity χ_i will take some number, t, of distinct values; and the powers of S, which are of order m, will belong to t, or a multiple of t, distinct conjugate sets, for which the t values of χ_i are the characteristics, each set containing the same number h_i of operations. Summing for these sets

$$\Sigma h_i \chi_i = h_i M_i s_m,$$

where s_m is the sum of the primitive m^{th} roots of unity, and M_i is some positive integer, not less than unity. If m is p^a, where p is a prime, then

$$s_m = \;\;0, \quad a > 1,$$

$$s_m = -1, \quad a = 1.$$

Hence, if the order of Γ is a power of p, the above sum, extended to the conjugate sets containing powers of S which are of the same order as itself, is

$$-h_i M_i,$$

when the order of S is p, and zero when the order of S is $p^a\,(a > 1)$. Now the characteristics of any irreducible representation of G satisfy the equation*

$$\Sigma h_i \chi_i = 0,$$

the sum being extended to all the conjugate sets. For the identical operation $\chi_1 = n$, the number of variables. Hence in the present case this equation takes the form

$$n - h_i M_i - h_j M_j - \ldots = 0.$$

Now† $h_i \chi_i / \chi_1$ is an algebraic integer, and therefore $h_i M_i / n$ is a rational integer. Hence

$$n = h_i M_i, \quad h_j = \ldots = 0.$$

Now a group whose order is the power of a prime necessarily has self-conjugate operations, so that h_i must be unity. The relations

$$h_j = \ldots = 0$$

implies that the group has no other operations of order p except the self-conjugate operation S and its powers; or in other words that G has only a single subgroup of order p. If p is an odd prime the group is then cyclical,‡ and n the number of variables is therefore unity. If p is 2 the group is either cyclical or of the type§ defined by

$$P^{2^{n-1}} = 1, \;\; Q^2 = P^{2^{n-2}}, \;\; Q^{-1} P Q = P^{-1}.$$

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* "On group characteristics," *Proc. L.M.S*, Vol. XXXIII., p. 153.
† "On groups of order $p^a q^\beta$," *Proc. L.M.S*, New Series, Vol. I., p. 389.
‡ *Theory of Groups*, p. 73.
§ *Theory of Groups*, p. 75.

It is therefore necessary to determine the irreducible groups of linear substitutions which are simply isomorphic with a group of this type.

The order of the group is 2^n. Those of its operations which do not belong to the subgroup generated by P fall into the two conjugate sets,

$$Q,\ QP^2,\ QP^4,\ \ldots,$$
$$QP,\ QP^3,\ QP^5,\ \ldots.$$

The operations of the subgroup generated by P are conjugate each with its inverse, $P^{2^{n-2}}$ and identity being self-conjugate operations. The number r of conjugate sets is therefore $2^{n-2}+3$. The derived group is that generated by P^2 of order 2^{n-2}; so that the number of irreducible representations of the group in a single variable is 4.

Now $$\chi_1^{\,1},\ \chi_1^{\,2},\ \ldots\ \chi_1^{\,r},$$

denoting the number of variables in the r distinct irreducible representations of the group,* and N being the order

$$N=\sum_1^r (\chi_1^{\,k})^2.$$

Here $N=2^n$, four of the χ's are unity, and r is $2^{n-2}+3$. Hence

$$2^n-4=\text{sum of } 2^{n-2}-1 \text{ square integers};$$

and therefore each other χ_1 must be 2. The group can therefore only be represented as an irreducible group in 2 variables. If the representation is simply isomorphic with the abstract group, P must be a substitution of order 2^{n-1} with a self-inverse characteristic. Hence P may be taken to be

$$x'=\alpha x,\quad y'=\alpha^{-1}y,$$

where α is a primitive 2^{n-1}th root of unity. The most general substitution of order 4 which transforms P into its inverse is

$$x'=ky,\quad y'=-\frac{1}{k}x,$$

and on replacing the variables by suitable multiples of themselves the k may be made unity. Hence the only distinct

* "On group characteristics," *loc. cit.*, p. 153.

representations of the group in two variables, which are simply isomorphic with it, consist of the group generated by

$$P \sim x' = \alpha x, \quad y' = \alpha^{-1} y,$$

$$Q \sim x' = y, \quad y' = -x,$$

and those that arise from this on replacing α by another primitive 2^{n-1}th root of unity.

Hence, if the order of Γ is the power of a prime, the only exceptions are

(i) a cyclical group in a single variable; and

(ii) a group of order 2^n in 2 variables, which contains only one operation of order **2**.

3. Consider next the general case, where the order of Γ, supposed not to possess the property, is not the power of a prime. Let p^α be the highest power of an odd prime p that divides the order of Γ. A subgroup of Γ of order p^α will in general be reducible, and for each irreducible component there must be no subgroup which has a linear invariant. Hence each irreducible component is a cyclical group in a single variable.

If the cyclical component generated by

$$x' = \omega x$$

were not simply isomorphic with the subgroup of order p^α, then x would be a linear invariant for some subgroup of order p^a $(0 < a < \alpha)$. Hence the subgroups of order p^a are cyclical. Similarly it may be shown that if the order of Γ is even, and 2^β is the greatest power of 2 which divides it, then the subgroups of order 2^a are necessarily either cyclical or of the type given at the end of the last paragraph. Moreover in the latter case each irreducible component of a subgroup of order 2^β must transform 2 variables among themselves, and all the irreducible components must be simply isomorphic.

4. When the order of Γ is not the power of a prime there are then just three classes of cases in which it may happen that no subgroup of Γ has a linear invariant.

(i) In the first case all the Sylow subgroups must be cyclical. The representation of any such group as an irreducible group of linear substitutions has been considered in

the paper which immediately precedes this; and the conditions determined under which no substitution of such a group has an unit multiplier.

(ii) In the second case all the Sylow subgroups of odd order are cyclical, while there are non-cyclical subgroups of order 2^a, which contain only one operation of order 2. Moreover any operation of the group which is permutable with a subgroup of order 2^a is permutable with every operation of that subgroup. These groups, like the previous ones, are soluble; and it is shown in the previous paper that exceptions of this class actually occur.

(iii) In the third case the order of Γ is of the form $2^3 3^a p_3^{\beta}, \ldots, (\alpha > 0)$. The subgroups of order 2^3 are quaternion-groups, and each is contained in a subgroup of order 24, in which the six operations of order 4 form a single conjugate set. Such a group of order 24 is simply isomorphic with the irreducible group in two variables generated by

$$x' = \quad ix; \quad x' = \quad y; \quad x' = \tfrac{1}{2}(i-1)x + \tfrac{1}{2}(i+1)y;$$

$$y' = -iy; \quad y' = -x; \quad y' = \tfrac{1}{2}(i-1)x - \tfrac{1}{2}(i+1)y;$$

and no subgroup of this has a linear invariant. Moreover a subgroup of this type is contained in the group of order 120, which is generated by

$$x' = \epsilon x; \quad \sqrt{5}x' = -(\epsilon - \epsilon^4)x + (\epsilon^2 - \epsilon^3)y;$$

$$y' = \epsilon^4 y; \quad \sqrt{5}y' = \quad (\epsilon^2 - \epsilon^3)x + (\epsilon - \epsilon^4)y;$$

where ϵ is a 5th root of unity. This group, which is known to be isomorphic with the icosahedral group in respect of its self-conjugate subgroup of order 2, is not soluble, and has no subgroup for which a linear function of x and y is invariant. This third class of cases again then contains actual exceptions to the rule, and they are not as in the former cases necessarily soluble.

Subject then to these three very limited classes of exceptions, the general theorem holds that:—An irreducible finite group of linear substitutions has substitutions, other than the identical one, one or more of whose multipliers are unity.

THE DEFINITION OF A SERIES SIMILARLY ORDERED TO THE SERIES OF ALL ORDINAL NUMBERS.

By *Philip E. B. Jourdain*, Trinity College, Cambridge.

I HAVE proved* that every aggregate for which a cardinal number exists (in the mathematical sense of the word) must be capable of arrangement in a well-ordered series which is similarly ordered to some *segment* of the well-ordered series (W) of all ordinal numbers. Although we cannot, without contradiction, define ordinal *numbers* which transcend all the numbers of W, there is no reason against defining other well-ordered series having a series similar to W as a mere segment; and, accordingly, W is similar to a segment merely of a series ($\mathfrak{W}$)† such that every well-ordered series is similar either to it or to a segment of it. Since $\mathfrak{W}$ can, without difficulty, be proved to be well-ordered,‡ the definition of a series similar to $\mathfrak{W}$ followed by an element or elements is, at once, seen to be contradictory. Thus, $\mathfrak{W}$ may appropriately be called an *absolutely* infinite series, and any finite or transfinite aggregate can be represented on a segment of $\mathfrak{W}$; and every segment has a cardinal and ordinal number when, and only when, it is also representable on a segment of W. As for W itself, since it is only representable on a segment of $\mathfrak{W}$, it can only be called 'absolutely' infinite if, by definition, such a term is applied to any aggregate which has no cardinal number.§

Since, then, the series W gives us a theoretical criterion for the decision whether any given aggregate has a cardinal number or not, and since, also, we ought to be able to judge (at least theoretically) whether any given aggregate has a cardinal number, or any given series a type, before we can define the ordinal numbers, which make up the series W, it becomes important to define a series, which is not composed

* "On the Transfinite Cardinal Numbers of Well-ordered Aggregates," *Phil. Mag.*, Jan. 1904, pp. 61-75 (see especially pp. 61-67), "On Transfinite Cardinal Numbers of the Exponential Form," *ibid.*, Jan. 1905, pp. 42-56 (see especially pp. 51-56). In this second paper is pointed out the error I, in common with Schoenflies, made in the first (p. 67) in supposing that W is ordinally similar to $\mathfrak{W}$.

† Such a $\mathfrak{W}$ is evidently not unique, so that we cannot speak of 'the' series $\mathfrak{W}$. I choose any of the $\mathfrak{W}$'s and speak (for shortness) of that as 'the' $\mathfrak{W}$.

‡ Cf. the method of *Phil. Mag.*, Jan. 1904, pp. 65-66.

§ As is done in *Phil. Mag.*, Jan. 1905, p. 51.

of numbers (since this series is to serve to decide whether numbers are possible or not), similarly ordered to W.*

1. Now, a characteristic property of ordinal numbers is that

$$\omega_\gamma > \gamma \quad \text{..........................(1)},$$

where γ is any ordinal number, and ω_γ is the first number, or 'class-characteristic,' of that number-class such that the cardinal number of all the ordinal numbers less than ω_γ is

$$\aleph_\gamma\dagger.$$

The inequality (1) holds for all ordinal numbers γ, but if W had (which it has not) an ordinal number β, and a cardinal number, which would be

$$\aleph_\beta,$$

we should have, as I have proved,‡

$$\beta = \omega_\beta.$$

To define, then, the series required, we have to define a well-ordered aggregate (W) which falls into certain 'classes,' corresponding to the number-classes of W, such that, if we consider any element (a) of W (corresponding to γ), and that element (A) of W (corresponding to ω_γ) such that (1): A is the first element of a class; (2) the series of all the first elements of W which precede A is ordinally similar to the series of elements preceding a; then a is contained in a *segment* of the series bounded by A.

2. The 'classes' in W may be thus defined.

The series W is well-ordered and transfinite, and, in the further determination of W, we will only consider such terms of it as have an infinity of predecessors; that is to say, these predecessors must form an aggregate of equal power (Mächtigkeit) with a part of itself. In this transfinite part of W, which consists of all the terms subsequent to some definite one (corresponding to ω of W), every term is such that either the aggregate of terms preceding it is of equal power with the aggregate of terms preceding *some* term which precedes it, or this is not the case. In the latter case, the term in question marks the beginning of a 'class,' and is said to be a class-

* Cf. *Phil. Mag.*, Jan. 1904, p. 67.
† *Ibid.*, Jan. 1905, p. 53.
‡ *Ibid.*, pp. 52-53.

characteristic. If a and b are two neighbouring class-characteristics (it is to be noted that every term of a well-ordered series has at least one neighbour), and a precedes b, the series beginning with a and preceding b is a 'class.' The first class-characteristics correspond to the numbers

$$\omega, \omega_1, \omega_2, \ldots, \omega_\nu, \ldots\omega_\omega, \omega_{\omega+1}, \ldots, \ldots\omega_\gamma \ldots\ldots\ldots(2)$$

in W, but the series extends further than (2). For the first number in $\mathfrak{W}$ after all the terms which correspond to the terms of W is the first of a 'class.' Thus the conception of 'class' is a generalisation of Cantor's conception of 'number-class.'

3. The definition required is, then, by the two preceding sections, any well-ordered transfinite series W such that, if a is any element of W, and A is that class-characteristic corresponding to a when the series bounded by a is represented on the series of class-characteristics of W, then a is contained in a segment of the series bounded by A.

The Manor House, Broadwindsor,
Beaminster, Dorset,
March 20th, 1905.

ON THE SERIES FOR THE SINE AND COSINE.

By *M. J. M. Hill, M.A., Sc.D., F.R.S.*,

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THE series for $\sin x$ and $\cos x$ in powers of x are so fundamental that any simplification in their demonstration is not without importance. A comparison of the proofs in treatises on Trigonometry which are now accepted as valid with those of an earlier date, which were formerly thought sufficient, will show the complexity of the ideas on which modern demonstrations are based, although these ideas are now treated as elementary.

The object of this paper is to give an inductive demonstration of the series, that is to say a method by which any term is obtained from a knowledge of the preceding terms.

It is based on an equation given by Le Cointe in his *Théorie des Fonctions Circulaires* and quoted in Todhunter's *Trigonometry*, viz.

$$3^n \sin\frac{x}{3^n} - \sin x = 4\left(\sin^3\frac{x}{3} + 3\sin^3\frac{x}{3^2} + \ldots + 3^{n-1}\sin^3\frac{x}{3^n}\right).$$

Le Cointe himself, starting from the inequality

$$\sin x < x,$$

proved by using this on the right-hand side of the equation that it would follow that

$$\sin x > x - \frac{x^3}{3!}.$$

But he did not apparently continue the process, possibly for want of the theorem given in Art. 1 of this paper.

Now an examination of the form of his equation suggests that the process will succeed if continued.

For if the value of $\sin x$ be obtained up to the term containing x^{2r+1}, then, since the series for $\sin x$ begins with x, the terms obtained will give the value of $\sin^2 x$ up to x^{2r+2}, and therefore $\sin^3 x$ up to x^{2r+3}.

Making use of these terms in the right-hand side of Le Cointe's equation, the value of $\sin x$ which occurs on the left is given up to x^{2r+3}.

The proof depends on the following deductions from Le Cointe's equation

$$3^n \left\{\sin \frac{x}{3^n} - \frac{x}{3^n} + \frac{1}{3!}\left(\frac{x}{3^n}\right)^3 - \ldots - \frac{1}{(4r+5)!}\left(\frac{x}{3^n}\right)^{4r+5}\right\}$$

$$> \left(\sin x - x + \frac{1}{3!}x^3 - \ldots - \frac{1}{(4r+5)!}x^{4r+5}\right);$$

$$3^n \left\{\sin \frac{x}{3^n} - \frac{x}{3^n} + \frac{1}{3!}\left(\frac{x}{3^n}\right)^3 - \ldots + \frac{1}{(4r+7)!}\left(\frac{x}{3^n}\right)^{4r+7}\right\}$$

$$< \left(\sin x - x + \frac{1}{3!}x^3 - \ldots + \frac{1}{(4r+7)!}x^{4r+7}\right);$$

in each of which it should be noted that if the second member of either inequality is denoted by $\phi(x)$, then its first member will be $3^n\phi\left(\frac{x}{3^n}\right)$.

Before proceeding to the proof of the above propositions three lemmas will be given (Arts. 1, 2, 3).

ART. 1. *Lemma* 1. Let

$$u_1, u_3, \ldots, u_{4r+3}, u_{4r+5};$$

and

$$v_1, v_3, \ldots, v_{4r+3}, v_{4r+5};$$

be two series of positive magnitudes, each in descending order of magnitude.

Let U be greater than

$$(u_1 - u_3 + u_5 - \ldots + u_{4r+1} - u_{4r+3}),$$

but less than

$$(u_1 - u_3 + u_5 - \ldots + u_{4r+1} - u_{4r+3} + u_{4r+5}).$$

Let V be greater than

$$(v_1 - v_3 + v_5 - \ldots + v_{4r+1} - v_{4r+3}),$$

but less than

$$(v_1 - v_3 + v_5 - \ldots + v_{4r+1} - v_{4r+3} + v_{4r+5}).$$

Then it will be shown that UV is greater than

$$\begin{aligned}
& u_1 v_1 \\
& - (u_1 v_3 + u_3 v_1) \\
& + (u_1 v_5 + u_3 v_3 + u_5 v_1) \\
& + \ldots\ldots\ldots\ldots\ldots\ldots \\
& + (u_1 v_{4r+1} + u_3 v_{4r-1} + \ldots + u_{4r+1} v_1) \\
& - (u_1 v_{4r+3} + u_3 v_{4r+1} + \ldots + u_{4r+3} v_1),
\end{aligned}$$

but less than

$$\begin{aligned}
& u_1 v_1 \\
& - (u_1 v_3 + u_3 v_1) \\
& + (u_1 v_5 + u_3 v_3 + u_5 v_1) \\
& - \ldots\ldots\ldots\ldots\ldots\ldots \\
& + (u_1 v_{4r+1} + u_3 v_{4r-1} + \ldots + u_{4r+1} v_1) \\
& - (u_1 v_{4r+3} + u_3 v_{4r+1} + \ldots + u_{4r+3} v_1) \\
& + (u_1 v_{4r+5} + u_3 v_{4r+3} + \ldots + u_{4r+5} v_1).
\end{aligned}$$

To prove the first of these statements we have UV greater than the product

$$\begin{aligned}
& (v_1 - v_3 + v_5 - \ldots + v_{4r+1} - v_{4r+3}) \\
\times\ & (u_1 - u_3 + u_5 - \ldots + u_{4r+1} - u_{4r+3}).
\end{aligned}$$

We will multiply the upper series by the separate terms of the lower series, and then separate out the result into two parts, viz.

(*a*) Those terms in which the sum of the suffixes does not exceed $4r+4$.

These are exactly the terms whose aggregate is to be proved less than UV.

(*b*) Those terms in which the sum of the suffixes exceeds $4r+4$.

The aggregate of these is

$$\begin{aligned} & u_3 v_{4r+3} \\ & + u_5 (v_{4r+1} - v_{4r+3}) \\ & + u_7 (v_{4r-1} - v_{4r+1} + v_{4r+3}) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + u_{4r+3} (v_3 - v_5 + \ldots + v_{4r+3}). \end{aligned}$$

Each bracket containing the v-terms has a positive sum, and therefore the aggregate of these terms is positive. Hence UV is greater than the sum of the two groups of terms.

Since the aggregate of the second group is positive, it follows that UV is greater than the aggregate of the first group.

To prove the second statement we have UV less than the product

$$\begin{aligned} & (v_1 - v_3 + v_5 - \ldots + v_{4r+1} - v_{4r+3} + v_{4r+5}) \\ \times & (u_1 - u_3 + u_5 - \ldots + u_{4r+1} - u_{4r+3} + u_{4r+5}). \end{aligned}$$

Proceeding in like manner and separating the product into two parts, viz.

(*a*) Those terms in which the sum of the suffixes does not exceed $4r+6$.

These are exactly the terms whose aggregate is to be proved greater than UV.

(*b*) Those terms in which the sum of the suffixes exceeds $4r+6$.

The aggregate of these is

$$\begin{aligned} & - u_3 v_{4r+5} \\ & - u_5 (v_{4r+3} - v_{4r+5}) \\ & - u_7 (v_{4r+1} - v_{4r+3} + v_{4r+5}) \\ & - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & - u_{4r+5} (v_3 - v_5 + \ldots - v_{4r+5}), \end{aligned}$$

and is therefore negative.

Now UV is less than the aggregate of the two groups of terms.

The aggregate of the second group is negative.

Therefore UV is less than the aggregate of the first group.

ART. 2. *Lemma* 2. If

$$t_r = \frac{x^r}{r!},$$

then
$$t_1 t_{2s+1} + t_3 t_{2s-1} + t_5 t_{2s-3} + \ldots + t_{2s+1} t_1 = \frac{1}{2}\,\frac{(2x)^{2s+2}}{(2s+2)!},$$

$$t_1 t_{2s+1} + t_3 t_{2s-1} + t_5 t_{2s-3} + \ldots + t_{2s+1} t_1$$

$$= x^{2s+2}\left(\frac{1}{1!}\,\frac{1}{(2s+1)!} + \frac{1}{3!}\,\frac{1}{(2s-1)!} + \frac{1}{5!}\,\frac{1}{(2s-3)!} + \ldots + \frac{1}{(2s+1)!}\,\frac{1}{1!}\right)$$

$$= \frac{x^{2s+2}}{(2s+2)!}\left({}_{2s+2}C_1 + {}_{2s+2}C_3 + {}_{2s+2}C_5 + \ldots + {}_{2s+2}C_{2s+1}\right)$$

$$= \frac{x^{2s+2}}{(2s+2)!}\,\frac{1}{2}\left[(1+1)^{2s+2} - (1-1)^{2s+2}\right]$$

$$= \frac{1}{2}\,\frac{(2x)^{2s+2}}{(2s+2)!}.$$

ART. 3. *Lemma* 3. If

$$t_r = \frac{x^r}{r!}, \text{ and } w_r = \frac{(2x)^r}{r!},$$

then
$$t_1 w_{2s} + t_3 w_{2s-2} + t_5 w_{2s-4} + \ldots + t_{2s-1} w_2 = \frac{1}{2}\,\frac{x^{2s+1}}{(2s+1)!}\left(3^{2s+1} - 3\right).$$

The expression to be summed

$$= \frac{x}{1!}\,\frac{(2x)^{2s}}{(2s)!} + \frac{x^3}{3!}\,\frac{(2x)^{2s-2}}{(2s-2)!} + \frac{x^5}{5!}\,\frac{(2x)^{2s-4}}{(2s-4)!} + \ldots + \frac{x^{2s-1}}{(2s-1)!}\,\frac{(2x)^2}{2!}$$

$$= \frac{x^{2s+1}}{(2s+1)!}\left[2^{2s}\,{}_{2s+1}C_1 + 2^{2s-2}\,{}_{2s+1}C_3 + 2^{2s-4}\,{}_{2s+1}C_5 + \ldots + 2^2\,{}_{2s+1}C_{2s-1}\right]$$

$$= \frac{x^{2s+1}}{(2s+1)!}\,\frac{1}{2}\left[(2+1)^{2s+1} - (2-1)^{2s+1} - 2\right]$$

$$= \frac{1}{2}\,\frac{x^{2s+1}}{(2s+1)!}\left(3^{2s+1} - 3\right).$$

ART. 4. Starting from Le Cointe's equation

$$3^n \sin\frac{x}{3^n} - \sin x = 4\left(\sin^3\frac{x}{3} + 3\sin^3\frac{x}{3^2} + \dots + 3^{n-1}\sin^3\frac{x}{3^n}\right),$$

and using the property $\sin x < x$ on the right-hand side, it follows that

$$3^n \sin\frac{x}{3^n} - \sin x < 4\left[\left(\frac{x}{3}\right)^3 + 3\left(\frac{x}{3^2}\right)^3 + \dots + 3^{n-1}\left(\frac{x}{3^n}\right)^3\right].$$

Here it must be noticed that the excess of the second member of the inequality over the first is a finite magnitude, which, though depending on n, does not tend to zero when n is made infinite; for the difference is made up of the differences

$$\left(\frac{x}{3}\right)^3 - \sin^3\frac{x}{3}, \quad 3\left(\frac{x}{3^2}\right)^3 - 3\sin^3\frac{x}{3^2},$$

and so on.

The above inequality gives

$$3^n \sin\frac{x}{3^n} - \sin x < \frac{x^3}{3!}\left(1 - \frac{1}{3^{2n}}\right),$$

and this may be written

$$3^n\left\{\sin\frac{x}{3^n} - \frac{x}{3^n} + \frac{1}{3!}\left(\frac{x}{3^n}\right)^3\right\}$$
$$< \left(\sin x - x + \frac{x^3}{3!}\right).$$

Let the excess of the right-hand side over the left-hand side be the positive quantity δ.

Then

$$\sin x - x + \frac{x^3}{3!} = \delta + 3^n\left\{\sin\frac{x}{3^n} - \frac{x}{3^n} + \frac{1}{3!}\left(\frac{x}{3^n}\right)^3\right\}.$$

As has been remarked, δ is finite and does not tend to zero as n increases without limit.

The value of

$$3^n\left\{\sin\frac{x}{3^n} - \frac{x}{3^n} + \frac{1}{3!}\left(\frac{x}{3^n}\right)^3\right\}$$

tends to zero as n increases without limit. Hence n can be taken so large that

$$\delta + 3^n\left\{\sin\frac{x}{3^n} - \frac{x}{3^n} + \frac{1}{3!}\left(\frac{x}{3^n}\right)^3\right\}$$

is a finite positive quantity.

Therefore $$\sin x - x + \frac{x^3}{3!} > 0,$$

therefore $$\sin x > x - \frac{x^3}{3!}.$$

Starting from $$\sin x > x - \frac{x^3}{3!},$$

and supposing $x^2 < 6$, it follows that

$$\sin^2 x > x^2 - \frac{x^4}{3} + \frac{x^6}{36},$$

therefore $$\sin^2 x > x^2 - \frac{x^4}{3}.$$

If we now suppose $x^2 < 3$, it will follow that

$$\sin^3 x > x^3 - \frac{x^5}{2} + \frac{x^7}{18},$$

therefore $$\sin^3 x > x^3 - \frac{x^5}{2}.$$

Making use of this on the right-hand side of Le Cointe's equation, it follows that

$$3^n \sin\frac{x}{3^n} - \sin x$$

$$> 4\left[\left(\frac{x}{3}\right)^3 - \tfrac{1}{2}\left(\frac{x}{3}\right)^5 + 3\left\{\left(\frac{x}{3^2}\right)^3 - \tfrac{1}{2}\left(\frac{x}{3^2}\right)^5\right\} + \dots + 3^{n-1}\left\{\left(\frac{x}{3^n}\right)^3 - \tfrac{1}{2}\left(\frac{x}{3^n}\right)^5\right\}\right],$$

i.e., $$> \frac{x^3}{3!}\left(1 - \frac{1}{3^{2n}}\right) - \frac{x^5}{5!}\left(1 - \frac{1}{3^{4n}}\right).$$

This can be written

$$3^n\left\{\sin\frac{x}{3^n} - \frac{x}{3^n} + \frac{1}{3!}\left(\frac{x}{3^n}\right)^3 - \frac{1}{5!}\left(\frac{x}{3^n}\right)^5\right\}$$

$$> \left(\sin x - x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5\right).$$

Reasoning as before, it follows that

$$\sin x < x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5.$$

Starting from $\quad \sin x < x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$,

it follows that $\quad 2\sin^2 x < \dfrac{(2x)^2}{2!} - \dfrac{(2x)^4}{4!} + \dfrac{(2x)^6}{6!}$,

and $\quad 4\sin^3 x < (3^3-3)\dfrac{x^3}{3!} - (3^5-3)\dfrac{x^5}{5!} + (3^7-3)\dfrac{x^7}{7!}$.

Using the last result in the right-hand side of Le Cointe's equation, it follows that

$$3^n \sin\frac{x}{3^n} - \sin x$$

$$< \left\{\begin{aligned} &+(3^3-3)\frac{1}{3!}\left(\frac{x}{3}\right)^3 - (3^5-3)\frac{1}{5!}\left(\frac{x}{3}\right)^5 + (3^7-3)\frac{1}{7!}\left(\frac{x}{3}\right)^7 \\ &+(3^3-3)\frac{1}{3!}3\left(\frac{x}{3^2}\right)^3 - (3^5-3)\frac{1}{5!}3\left(\frac{x}{3^2}\right)^5 + (3^7-3)\frac{1}{7!}3\left(\frac{x}{3^2}\right)^7 \\ &+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &+(3^3-3)\frac{1}{3!}3^{n-1}\left(\frac{x}{3^n}\right)^3 - (3^5-3)\frac{1}{5!}3^{n-1}\left(\frac{x}{3^n}\right)^5 \\ &\qquad\qquad\qquad\qquad + (3^7-3)\frac{1}{7!}3^{n-1}\left(\frac{x}{3^n}\right)^7 \end{aligned}\right\},$$

i.e., $\quad < \dfrac{x^3}{3!}\left(1-\dfrac{1}{3^{2n}}\right) - \dfrac{x^5}{5!}\left(1-\dfrac{1}{3^{4n}}\right) + \dfrac{x^7}{7!}\left(1-\dfrac{1}{3^{6n}}\right)$.

This can be written

$$3^n\left\{\sin\frac{x}{3^n} - \frac{x}{3^n} + \frac{1}{3!}\left(\frac{x}{3^n}\right)^3 - \frac{1}{5!}\left(\frac{x}{3^n}\right)^5 + \frac{1}{7!}\left(\frac{x}{3^n}\right)^7\right\}$$

$$< \left(\sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!}\right).$$

Reasoning as before it follows that

$$\sin x > x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

ART. 5. It is now possible to make an induction. Suppose that it has been proved that

$$\sin x > \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{x^{4r+1}}{(4r+1)!} - \frac{x^{4r+3}}{(4r+3)!}\right),$$

the excess of the left-hand side over the right-hand side being a finite magnitude.

The terms are in diminishing order of magnitude if $x^2 < 6$.

Using the notation $t_r = \dfrac{x^r}{r!}$, we have

$$\sin x > t_1 - t_3 + t_5 - \ldots + t_{4r+1} - t_{4r+3}.$$

Hence, applying Art. 1, and putting

$$u_s = v_s = t_s,$$

it follows that

$$\sin^2 x > \left\{\begin{array}{l} \quad t_1^2 \\ -(t_1t_3 + t_3t_1) \\ +(t_1t_5 + t_3t_3 + t_5t_1) \\ - \ldots\ldots\ldots\ldots\ldots\ldots \\ +(t_1t_{4r+1} + t_3t_{4r-1} + \ldots + t_{4r+1}t_1) \\ -(t_1t_{4r+3} + t_3t_{4r+1} + \ldots + t_{4r+3}t_1). \end{array}\right\}.$$

Hence, by Art. 2.

$$2\sin^2 x > \frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \ldots + \frac{(2x)^{4r+2}}{(4r+2)!} - \frac{(2x)^{4r+4}}{(4r+4)!},$$

or, putting $$w_s = \frac{(2x)^s}{s!},$$

$$2\sin^2 x > w_2 - w_4 + \ldots + w_{4r+2} - w_{4r+4}.$$

The terms will be in diminishing order of magnitude if $x^2 < 3$.

In this case we again apply Art. 1, taking

$$u_r = t_r,\ v_r = w_{r+1}.$$

It follows that

$$2\sin^2 x > \left\{\begin{array}{l} \quad t_1w_2 \\ -(t_1w_4 + t_3w_2) \\ + \ldots\ldots\ldots\ldots\ldots \\ +(t_1w_{4r+2} + t_3w_{4r} + \ldots + t_{4r-1}w_2) \\ -(t_1w_{4r+4} + t_3w_{4r+2} + \ldots + t_{4r+3}w_2) \end{array}\right\}.$$

Hence, using the result of Art. 3, it follows that

$$4\sin^3 x > \frac{x^3}{3!}(3^3-3) - \frac{x^5}{5!}(3^5-3) + \dots$$
$$+ \frac{x^{4r+3}}{(4r+3)!}(3^{4r+3}-3) - \frac{x^{4r+5}}{(4r+5)!}(3^{4r+5}-3).$$

Using this result on the right-hand side of Le Cointe's equation, it follows that

$$3^n \sin\frac{x}{3^n} - \sin x$$

$$> \frac{3^3-3}{3!}x^3\left\{\frac{1}{3^3} + 3\left(\frac{1}{3^2}\right)^3 + 3^2\left(\frac{1}{3^3}\right)^3 + \dots + 3^{n-1}\left(\frac{1}{3^n}\right)^3\right\}$$
$$- \frac{3^5-3}{5!}x^5\left\{\frac{1}{3^5} + 3\left(\frac{1}{3^2}\right)^5 + 3^2\left(\frac{1}{3^3}\right)^5 + \dots + 3^{n-1}\left(\frac{1}{3^n}\right)^5\right\}$$
$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$+ \frac{3^{4r+3}-3}{(4r+3)!}x^{4r+3}\left\{\frac{1}{3^{4r+3}} + 3\left(\frac{1}{3^2}\right)^{4r+3} + 3^2\left(\frac{1}{3^3}\right)^{4r+3} + \dots + 3^{n-1}\left(\frac{1}{3^n}\right)^{4r+3}\right\}$$
$$- \frac{3^{4r+5}-3}{(4r+5)!}x^{4r+5}\left\{\frac{1}{3^{4r+5}} + 3\left(\frac{1}{3^2}\right)^{4r+5} + 3^2\left(\frac{1}{3^3}\right)^{4r+5} + \dots + 3^{n-1}\left(\frac{1}{3^n}\right)^{4r+5}\right\}$$

therefore $$3^n \sin\frac{x}{3^n} - \sin x$$

$$> \frac{x^3}{3!}\left(1 - \frac{1}{3^{2n}}\right) - \frac{x^5}{5!}\left(1 - \frac{1}{3^{4n}}\right) + \dots$$
$$+ \frac{x^{4r+3}}{(4r+3)!}\left(1 - \frac{1}{3^{(4r+2)n}}\right) - \frac{x^{4r+5}}{(4r+5)!}\left(1 - \frac{1}{3^{(4r+4)n}}\right).$$

This may be written in the form

$$3^n\left\{\sin\frac{x}{3^n} - \frac{x}{3^n} + \frac{1}{3!}\left(\frac{x}{3^n}\right)^3 - \dots\right.$$
$$\left. + \frac{1}{(4r+3)!}\left(\frac{x}{3^n}\right)^{4r+3} - \frac{1}{(4r+5)!}\left(\frac{x}{3^n}\right)^{4r+5}\right\}$$
$$> \left(\sin x - x + \frac{x^3}{3!} - \dots + \frac{x^{4r+3}}{(4r+3)!} - \frac{x^{4r+5}}{4r+5)!}\right);$$

or, if $\phi(x)$ be written for the second member of the inequality, then it becomes

$$3^n\phi\left(\frac{x}{3^n}\right) > \phi(x).$$

Now, since the excess of the first member of the inequality over the second member is finite, whilst the limit of the first member for $n=\infty$ is zero, it follows that the second member is less than zero by a finite magnitude. Therefore

$$\sin x < x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \ldots - \frac{x^{4r+3}}{(4r+3)\,!} + \frac{x^{4r+5}}{(4r+5)\,!}.$$

Starting from this inequality, it follows in like manner that

$$2\sin^2 x < \frac{(2x)^2}{2\,!} - \frac{(2x)^4}{4\,!} + \ldots - \frac{(2x)^{4r+4}}{(4r+4)\,!} + \frac{(2x)^{4r+6}}{(4r+6)\,!},$$

and

$$4\sin^3 x < (3^3-3)\frac{x^3}{3\,!} - (3^5-3)\frac{x^5}{5\,!} + \ldots$$

$$- (3^{4r+5}-3)\frac{x^{4r+5}}{(4r+5)\,!} + (3^{4r+7}-3)\frac{x^{4r+7}}{(4r+7)\,!}.$$

Using the last result in the right-hand side of Le Cointe's equation, it follows that

$$3^n\sin\frac{x}{3^n} - \sin x$$

$$< \left(1-\frac{1}{3^{2n}}\right)\frac{x^3}{3\,!} - \left(1-\frac{1}{3^{4n}}\right)\frac{x^5}{5\,!} + \ldots$$

$$- \left(1-\frac{1}{3^{(4r+4)n}}\right)\frac{x^{4r+5}}{(4r+5)\,!} + \left(1-\frac{1}{3^{(4r+6)n}}\right)\frac{x^{4r+7}}{(4r+7)\,!}.$$

Therefore

$$3^n\left\{\sin\frac{x}{3^n} - \frac{x}{3^n} + \frac{1}{3\,!}\left(\frac{x}{3^n}\right)^3 - \ldots\right.$$

$$\left. - \frac{1}{(4r+5)\,!}\left(\frac{x}{3^n}\right)^{4r+5} + \frac{1}{(4r+7)\,!}\left(\frac{x}{3^n}\right)^{4r+7}\right\}$$

$$< \left(\sin x - x + \frac{x^3}{3\,!} - \ldots - \frac{x^{4r+5}}{(4r+5)\,!} + \frac{x^{4r+7}}{(4r+7)\,!}\right);$$

or, denoting the second member of the inequality by $\psi(x)$, the inequality is of the form

$$3^n\psi\left(\frac{x}{3^n}\right) < \psi(x).$$

Reasoning as before it follows that

$$\sin x > x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{x^{4r+5}}{(4r+5)!} - \frac{x^{4r+7}}{(4r+7)!}.$$

This inequality can be obtained from the inequality assumed at the beginning of the article by changing r into $(r+1)$. Now we have proved in Art. 4 that the inequality assumed at the beginning of this article holds when $r=0$ and $r=1$, therefore it holds when $r=2$, and so on it holds universally for any positive integral value of r.

ART. 6. It is therefore proved that $\sin x$ is greater than any even number of terms of the series

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots,$$

reckoned from the beginning, but that it is less than any odd number of terms of the series, reckoned from the beginning, provided $x^2 < 3$.

Since the series is absolutely convergent, it follows that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots.$$

Again, since

$$2\sin^2 x = 1 - \cos 2x,$$

we have, from the values of $2\sin^2 x$ given in Art. 5,

$$\cos 2x < 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \ldots + \frac{(2x)^{4r+4}}{(4r+4)!},$$

but $$\cos 2x > 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \ldots + \frac{(2x)^{4r+4}}{(4r+4)!} - \frac{(2x)^{4r+6}}{(4r+6)!}.$$

This gives the series for $\cos x$.

The series for $\cos 2x$ has been proved for $x^2 < 3$, and therefore for $(2x)^2 < 12$.

Hence the series for $\cos x$ has been proved for $x^2 < 12$.

Hence the series for $\sin x$ and $\cos x$ are both proved for $x^2 < 3$, and therefore for x equal to any acute angle.

Both series are absolutely convergent, and therefore, after deducing the formulæ for $\sin(x+y)$ and $\cos(x+y)$ from the series, the series may be demonstrated for angles of any magnitude.

ON LINEAR HOMOGENEOUS DIFFERENCE EQUATIONS.

By *Alf Guldberg.*

PROFESSOR S. PINCHERLE* is the first who has introduced the notion of irreducibility in the theory of linear homogeneous difference equations. A linear homogeneous difference equation with rational coefficients is, according to the terminology of Pincherle, called *irreducible*, if the equation has no integral common with a linear homogeneous difference equation of lower order with rational coefficients. If a given linear homogeneous difference equation with rational coefficients has an integral common with a linear homogeneous difference equation of lower order with rational coefficients it is called *reducible.*

Pincherle shows that a linear homogeneous difference equation with rational coefficients which has an integral common with an irreducible linear homogeneous difference equation with rational coefficients admits of all its integrals.

I propose in this note to indicate a method according to which it may be possible to decide whether a given linear homogeneous difference equation with rational coefficients is irreducible or not.

Let $y_x^{(1)}, y_x^{(2)}, \dots, y_x^{(n)}$ be a fundamental system of integrals of the linear homogeneous difference equation

$$y_{x+n} + p_x^{(1)} y_{x+n-1} + p_x^{(2)} y_{x+n-2} + \dots + p_x^{(n)} y_x = 0 \dots\dots (1),$$

where the p_x's are rational functions of x.

If the equation (1) is reducible there exists another linear homogeneous difference equation with rational coefficients

$$y_{x+m} + q_x^{(1)} y_{x+m-1} + q_x^{(2)} y_{x+m-2} + \dots + q_x^{(m)} y_x = 0 \quad (m < n) \dots\dots (2),$$

which has all its integrals common with (1). Let

$$y_x^{(1)}, \; y_x^{(2)}, \; \dots, \; y_x^{(m)}$$

be a fundamental system of integrals of (2). Hence, if we put

$$\Delta_x^{(m)} = \begin{vmatrix} y_x^{(1)}, & y_{x+1}^{(1)}, & \dots, & y_{x+m-1}^{(1)} \\ y_x^{(2)}, & y_{x+1}^{(2)}, & \dots, & y_{x+m-1}^{(2)} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ y_x^{(m)}, & y_{x+1}^{(m)}, & \dots, & y_{x+m-1}^{(m)} \end{vmatrix},$$

* See *S. Pincherle*: Le operazioni distributive et le loro applicazioni all' analisi, p. 235.

we have

$$\Delta_x^{(0)} q_x^{(i)} = (-1)^i \begin{vmatrix} y_x^{(1)}, y_{x+1}^{(1)}, \ldots, y_{x+i-1}^{(1)}, y_{x+m}^{(1)}, y_{x+i+1}^{(1)}, \ldots, y_{x+m-1}^{(1)} \\ y_x^{(2)}, y_{x+1}^{(2)}, \ldots, y_{x+i-1}^{(2)}, y_{x+m}^{(2)}, y_{x+i+1}^{(2)}, \ldots, y_{x+m-1}^{(2)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_x^{(m)}, y_{x+1}^{(m)}, \ldots, y_{x+i-1}^{(m)}, y_{x+m}^{(m)}, y_{x+i+1}^{(m)}, \ldots, y_{x+m-1}^{(m)} \end{vmatrix} \equiv (-1)^i \Delta_x^{(i)} \quad (i = 1.2\ldots m),$$

and particular
$$q_x^{(m)} = (-1)^m \frac{\Delta_{x+1}^{(0)}}{\Delta_x^{(0)}}.$$

The existence of the equation (2) requires then that the quotients $\frac{\Delta_x^{(i)}}{\Delta_x^{(0)}}$ are rational, and $\Delta_x^{(0)} = \Pi(-1)^m q^{(m)}$, and consequently $\Delta_x^{(i)}$ of the same form. A necessary condition for the reducibility of the equation (1) is then that the determinants of a system of $m < n$ linear independent integrals $y_x^{(1)}, y_x^{(2)}, \ldots, y_x^{(m)}$ of (1) for one of the values

$$m = 1, 2, \ldots, n-1$$

has the form
$$\Pi r(x),$$

where $r(x)$ is a rational function of x.

It is now easy to show that each determinant $\Delta_x^{(i)}$ satisfies a linear homogeneous difference equation with rational coefficients. We have only to form the successive values of the determinant $\Delta_x^{(i)}$, and in the expressions obtained to eliminate the higher successive values of y_x by means of y_{x+n}, given by the given equation (1), In this manner we get the successive values of $\Delta_x^{(i)}$ expressed as a linear aggregate of determinants

$$\Delta_x^{k_1 k_2 \ldots k_m} = \begin{vmatrix} y_{x+k_1}^{(1)}, & y_{x+k_2}^{(1)}, & \ldots, & y_{x+k_m}^{(1)} \\ y_{x+k_1}^{(2)}, & y_{x+k_2}^{(2)}, & \ldots, & y_{x+k_m}^{(2)} \\ \ldots & \ldots & \ldots & \ldots \\ y_{x+k_1}^{(m)}, & y_{x+k_2}^{(m)}, & \ldots, & y_{x+k_m}^{(m)} \end{vmatrix},$$

where $k_1, k_2, \ldots, k_m$ signify m arbitrary numbers of the series

0, 1, 2, ..., $m-1$. Denoting these determinants by $D_x^{(1)}, D_x^{(2)}, \ldots, D^{\binom{n}{m}}$ we get the following system

$$\Delta_{x+\rho}^{(i)} = a_x^{(\rho i1)} D_x^{(1)} + a_x^{(\rho i2)} D_x^{(2)} + \ldots a_x^{(\rho i\sigma)} D_x^{(\sigma)}, \; \sigma = \binom{n}{m}, \; \rho=1,2,\ldots\sigma,$$

where the a_x are rational functions of x.

From this system we get for $\Delta_x^{(i)}$ a linear homogeneous difference equation (E_i) of an order at highest $\binom{n}{m}$ with rational coefficients.

Let the equation obtained for $\Delta_x^{(0)}$ be

$$\Delta_{x+\sigma}^{(0)} + P_{(x)}^{(1)} \Delta_{x+\sigma-1}^{(0)} + \ldots + P_x^{(\sigma)} \Delta_x^{(0)} = 0 \;\ldots\ldots\ldots(3).$$

If the equation (2) exist, $q_x^{(m)} = (-1)^m \dfrac{\Delta_{x+1}^{(0)}}{\Delta_x}$ must be a rational function of x; the equation (3) must then have an integral y_x, where the quotient $\dfrac{y_{x+1}}{y_x}$ is rational.

In like manner we may form the different linear homogeneous difference equations (E_i), which are satisfied by the determinant $\Delta_x^{(i)}$, and examine whether they possess an integral y_x, when the quotient $\dfrac{y_{x+1}}{y_x}$ is rational.

Having examined this point the integrals of the equations (E_i) give the possible values of $\Delta_x^{(i)}$. We may then so associate them, that the quotients $\Delta_x^{(i)} : \Delta_x^{(0)}$ are rational; we have then finally to inquire whether the general integral of the linear homogeneous difference equations of the order m satisfies the given linear homogeneous difference equation.

The problem of the irreducibility of a given linear homogeneous difference equation is thus reduced to the following investigation: Examine whether a given linear homogeneous difference equation admits of an integral y_x, such that the quotient $\dfrac{y_{x+1}}{y_x}$ is rational—a problem which, at least in some cases, may be solved by the method of indeterminate coefficients.

Christiania,
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ON VARIOUS EXPRESSIONS FOR h, THE NUMBER OF PROPERLY PRIMITIVE CLASSES FOR A DETERMINANT $-p$, WHERE p IS A PRIME OF THE FORM $4n+3$.

(First Paper.)

By *H. Holden.*

1. Using the notation

$$4\frac{x^p-1}{x-1} = 4X_1X_2 = Y^2 + pZ^2,$$

$$H = \frac{h}{2-2/p},$$

the chief results obtained are

$$\left(\frac{dY}{dx}\right)_{x=1} = (-1)^{\frac{1}{2}(h-1)}\,.\,2/p\,.\,pH,$$

$$\left(\frac{d^2Y}{dx^2}\right)_{x=1} = (-1)^{\frac{1}{2}(h-1)}\,.\,2/p\,.\,\frac{p(p-3)}{2}\,.\,H,$$

$$\left(\frac{dZ}{dx}\right)_{x=-1} = -h,$$

$$\left(\frac{d^2Z}{dx^2}\right)_{x=-1} = \frac{(p-3)h}{2},$$

$$\Sigma\frac{1}{1-r^\alpha} - \Sigma\frac{1}{1-r^\beta} = i\sqrt{p}\,.\,H,\text{*}$$

$$\Sigma\frac{1}{1+r^\alpha} - \Sigma\frac{1}{1+r^\beta} = 2/p\,.\,i\sqrt{p}\,.\,h.$$

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* Schemmel (quoted in Bachmann's *Zahlentheorie*, Theil 2, pp. 227–230) has obtained this result by a different method. Putting $F(x) = \log\frac{\Pi\,(x-r^\beta)}{\Pi\,(x-r^\alpha)}$, he shows that $h = \frac{i\,(2-2/p)\,F'(1)}{\sqrt{p}} = -\frac{F'(i)}{\sqrt{p}} = \frac{F'(-i)}{\sqrt{p}}$. I have included this result in the present paper for the sake of the method used by me, which has led to further developments.

2. To prove that $\left(\dfrac{dZ}{dx}\right)_{x=-1} = -h.$

Writing $$X_1 = \Pi\,\{(x+1) - (1+r^{\alpha})\},$$
$$X_2 = \Pi\,\{(x+1) - (1+r^{\beta})\},$$

then, since $$4X_1X_2 = (X_1+X_2)^2 - (X_1-X_2)^2,$$

we should have, remembering that the first term of Z is generally given a positive sign,

$$\left(\frac{dZ}{dx}\right)_{x=-1} = -\frac{1}{i\sqrt{p}}$$

$\times$ coefficient of $(x+1)$ in the expansion of $X_1 - X_2$

$$= -\frac{1}{i\sqrt{p}}\left\{\Pi\,(1+r^{\alpha})\,\Sigma\,\frac{1}{1+r^{\alpha}} - \Pi\,(1+r^{\beta})\,\Sigma\,\frac{1}{1+r^{\beta}}\right\}.$$

If $2/p = +1$,

$$\Pi\,(1-r^{\alpha}) = \Pi\,(1-r^{2\alpha}),$$

therefore $$\Pi\,(1+r^{\alpha}) = 1,$$

and similarly $$\Pi\,(1+r^{\beta}) = 1.$$

If $2/p = -1$,

$$\Pi\,(1-r^{\alpha}) = \Pi\,(1-r^{-2\alpha})$$
$$= (-1)^{\frac{1}{2}(p-1)}\,r^{2\Sigma\alpha}\,\Pi\,(1-r^{2\alpha}),$$

therefore $$\Pi\,(1+r^{\alpha}) = \Pi\,(1+r^{\beta}) = -1,$$

or in both cases

$$\Pi\,(1+r^{\alpha}) = \Pi\,(1+r^{\beta}) = 2/p.^{*}$$

Again

$$\Sigma\,\frac{1}{1+r^{\alpha}} = \Sigma\,\frac{1-r^{\alpha}}{1-r^{2\alpha}} = \Sigma\,\frac{1-r^{(p+1)\alpha}}{1-r^{2\alpha}}$$
$$= \Sigma\,(1 + r^{2\alpha} + r^{4\alpha} + \ldots r^{(p-1)\alpha}),$$

and $$\Sigma\,\frac{1}{1+r^{\beta}} = \Sigma\,(1 + r^{2\beta} + r^{4\beta} + \ldots r^{(p-1)\beta}).$$

---

* A more general result is $\Pi\,\dfrac{1-r^{q\alpha}}{1-r^{\alpha}} = \Pi\,\dfrac{1-r^{q\beta}}{1-r^{\beta}} = q/p$, where p is a prime of form $4n+3$.

therefore

$$\Sigma \frac{1}{1+r^\alpha} - \Sigma \frac{1}{1+r^\beta} = 2/p\,(\Sigma r^\alpha - \Sigma r^\beta)\left(1/p + 2/p + \ldots \frac{p-1}{2}/p\right)$$

$$= 2/p \, . \, i\sqrt{p} \, . \, h^*$$

therefore $$\left(\frac{dZ}{dx}\right)_{x=-1} = -h.$$

3. To prove that $\left(\frac{dY}{dx}\right)_{x=1} = (-1)^{\frac{1}{2}(h-1)} . 2/p . pH.$

Writing $$X_1 = \Pi\,\{(x-1) + (1-r^\alpha)\},$$

$$X_2 = \Pi\,\{(x-1) + (1-r^\beta)\},$$

we have

$$\left(\frac{dY}{dx}\right)_{x=1} = \Pi\,(1-r^\alpha)\,\Sigma\frac{1}{1-r^\alpha} + \Pi\,(1-r^\beta)\,\Sigma\frac{1}{1-r^\beta}.$$

If $2/p = 1$,

$$\Pi\,(1-r^\alpha) = \Pi\,(1-r^{2\alpha})$$

$$= (-1)^{\frac{1}{2}(p-1)}\,\Pi r^\alpha\,\Pi\,(r^\alpha - r^{-\alpha})$$

$$= -\,\Pi\,(r^\alpha - r^{-\alpha})$$

$$= -\,(2i)^{\frac{1}{2}(p-1)}\,\Pi \sin\frac{2\alpha\pi}{p}$$

$$= 2^{\frac{1}{2}(p-1)}\,i\Pi \sin\frac{2\alpha\pi}{p},$$

since $$p = 8n+7,$$

and so the sign of $\Pi\,(1-r^\alpha) = (-1)^b$, where b is the number of quadratic residues between $\frac{p-1}{2}$ and p, or the number of non-residues less than $\frac{p}{2}$.

Therefore, if $2/p = 1$,

$$\Pi\,(1-r^\alpha) = -\,\Pi\,(1-r^\beta) = (-1)^b\,i\sqrt{p}.$$

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* This result is clearly true whether p be prime or the product of different primes.

If $2/p = -1$, the same values would be got, since

$$\Pi(1 - r^{2a}) = \Pi(1 - r^{\beta}),$$

and
$$-(2i)^{\frac{1}{2}(p-1)} = -2^{\frac{1}{2}(p-1)}i \text{ as } p = 8n+3.$$

These results may also be written

$$\Pi(1 - r^{a}) = -\Pi(1 - r^{\beta}) = (-1)^{\frac{1}{2}(h+1)}\, 2/p \,.\, i\sqrt{p}.$$

Again, if $2/p = 1$,

$$\Sigma \frac{1}{1 - r^{a}} = \Sigma \frac{1}{1 - r^{2a}},$$

and also equals

$$\Sigma \frac{1 + r^{a}}{1 - r^{2a}},$$

therefore
$$\Sigma \frac{r^{a}}{1 - r^{2a}} = 0,$$

and similarly
$$\Sigma \frac{r^{\beta}}{1 - r^{\beta}} = 0.$$

Therefore

$$\Sigma \frac{1}{1 - r^{a}} = \Sigma \frac{1}{1 - r^{2a}} = \Sigma \left(1 + r^{2a} + r^{4a} + \dots r^{(p-1)a} + \frac{r^{a}}{1 - r^{2a}}\right)$$

$$= \Sigma \left(1 + r^{a} + r^{2a} + \dots r^{\frac{(p-1)a}{2}} + \frac{r^{a}}{1 - r^{2a}}\right),$$

and similarly for
$$\Sigma \frac{1}{1 - r^{\beta}},$$

therefore

$$\Sigma \frac{1}{1 - r^{a}} - \Sigma \frac{1}{1 - r^{\beta}} = (\Sigma r^{a} - \Sigma r^{\beta}) \left(1/p + 2/p + \dots \frac{p-1}{2}/p\right)$$

$$= i\sqrt{p}\,.\,h.$$

If $2/p = -1$,

$$\Sigma \frac{1}{1 - r^{a}} = \Sigma \frac{1}{1 - r^{-2a}},$$

and also equals

$$-\Sigma \frac{r^{-\alpha}}{1-r^{-\alpha}} = -\Sigma \frac{r^{-\alpha}+r^{-2\alpha}}{1-r^{-2\alpha}},$$

therefore $\quad \Sigma \dfrac{1+r^{-\alpha}+r^{-2\alpha}}{1-r^{-2\alpha}} = 0 = \Sigma \dfrac{1+r^{-\beta}+r^{-2\beta}}{1-r^{-2\beta}}$,

Thus

$$3\Sigma \frac{1}{1-r^{\alpha}} = 3\Sigma \frac{1}{1-r^{-2\alpha}}$$

$$= \Sigma \left\{ 1 + \frac{r^{-2\alpha}}{1-r^{-2\alpha}} + \frac{1}{1-r^{-2\alpha}} \right.$$

$$\left. + 1 + r^{-2\alpha} + r^{-4\alpha} + \ldots + r^{(-p+1)\alpha} + \frac{r^{-\alpha}}{1-r^{-2\alpha}} \right\},$$

and similarly for $3\Sigma \dfrac{1}{1-r^{\beta}}$;

Therefore $\quad 3\left\{ \Sigma \dfrac{1}{1-r^{\alpha}} - \Sigma \dfrac{1}{1-r^{\beta}} \right\} = i\sqrt{(p)}h.$

Thus, whether $2/p = \pm 1$,

$$\Sigma \frac{1}{1-r^{\alpha}} - \Sigma \frac{1}{1-r^{\beta}} = i\sqrt{p}.H,^*$$

therefore $\quad \left(\dfrac{dY}{dx}\right)_{x=1} = (-1)^{\frac{1}{2}(h-1)}.2/p.pH.$

4. To prove that

$$\left(\frac{d^2 Y}{dx^2}\right)_{x=1} = (-1)^{\frac{1}{2}(h-1)} 2/p . \frac{p(p-3)}{2} . H,$$

$\left(\dfrac{d^2 Y}{dx^2}\right)_{x=1} = 2 \times$ coefficient of $(x-1)^2$ in the expansion of $X_1 + X_2$,

where $\quad X_1 = \Pi\{(x-1) + (1-r^{\alpha})\},$

$$X_2 = \Pi\{(x-1) + (1-r^{\beta})\},$$

---

* This result is clearly true for $p = 4n+3$, whether prime or the product of different primes.

therefore

$$\left(\frac{d^2Y}{dx^2}\right)_{x=1} = 2\left\{\Pi(1-r^\alpha)\,\Sigma\frac{1}{(1-r^{\alpha_1})(1-r^{\alpha_2})}\right.$$

$$\left. + \Pi(1-r^\beta)\,\Sigma\frac{1}{(1-r^{\beta_1})(1-r^{\beta_2})}\right\}$$

$$= 2\Pi(1-r^\alpha)\left\{\Sigma\frac{1}{(1-r^{\alpha_1})(1-r^{\alpha_2})} - \Sigma\frac{1}{(1-r^{\beta_1})(1-r^{\beta_2})}\right\}$$

$$= 2\Pi(1-r^\alpha)\,\tfrac{1}{2}\left\{\Sigma\left(\frac{1}{1-r^\alpha}\right)^2 - \Sigma\frac{1}{(1-r^\alpha)^2}\right.$$

$$\left. - \left(\Sigma\frac{1}{1-r^\beta}\right)^2 + \Sigma\frac{1}{(1-r^\beta)^2}\right\}.$$

But

$$\left(\Sigma\frac{1}{1-r^\alpha}\right)^2 - \left(\Sigma\frac{1}{1-r^\beta}\right)^2$$

$$= \left\{\Sigma\frac{1}{1-r^\alpha} + \Sigma\frac{1}{1-r^\beta}\right\}\left\{\Sigma\frac{1}{1-r^\alpha} - \Sigma\frac{1}{1-r^\beta}\right\}$$

$$= \frac{p-1}{2}\,.\,i\sqrt{(p)}H,$$

and $\quad \Sigma\dfrac{1}{(1-r^\alpha)^2} - \Sigma\dfrac{1}{(1-r^\beta)^2} = \Sigma\left\{\dfrac{1}{(1-r^\alpha)^2} - \dfrac{1}{(1-r^{-\alpha})^2}\right\}$

$$= \Sigma\left\{\left(\frac{1}{1-r^\alpha} + \frac{1}{1-r^{-\alpha}}\right)\left(\frac{1}{1-r^\alpha} - \frac{1}{1-r^{-\alpha}}\right)\right\}$$

$$= \Sigma\left(\frac{1}{1-r^\alpha} - \frac{1}{1-r^{-\alpha}}\right) = i\sqrt{(p)}\,H;$$

therefore $\quad \left(\dfrac{d^2Y}{dx^2}\right)_{x=1} = \Pi(1-r^\alpha)\,i\sqrt{(p)}\,H\left(\dfrac{p-1}{2} - 1\right)$

$$= 2\sqrt{p}.(-1)^{\frac{1}{2}(h-1)}.\frac{p(p-3)}{2}.H.$$

5. To prove that

$$\left(\frac{d^2Z}{dx^2}\right)_{x=-1} = \frac{(p-3)h}{2}$$

$\left(\dfrac{d^2Z}{dx^2}\right)_{x=-1} = \dfrac{2}{i\sqrt{p}} \times$ coefficient of $(x+1)^2$ in expansion of $X_1 - X_2$.

where $$X_1 = \Pi\{(x+1)-(1+r^{\alpha})\},$$
$$X_2 = \Pi\{(x+1)-(1+r^{\beta})\};$$

therefore

$$\left(\frac{d^2Z}{dx^2}\right)_{x=-1} = \frac{2}{i\sqrt{p}}\left\{\Pi(1+r^{\alpha})\Sigma\frac{1}{(1+r^{\alpha_1})(1+r^{\alpha_2})}\right.$$
$$\left. - \Pi(1+r^{\beta})\Sigma\frac{1}{(1+r^{\beta_1})(1+r^{\beta_2})}\right\}$$
$$= \frac{2}{i\sqrt{p}}\cdot\Pi(1+r^{\alpha}).\tfrac{1}{2}.\left\{\left(\Sigma\frac{1}{1+r^{\alpha}}\right)^2\right.$$
$$\left. - \Sigma\frac{1}{(1+r^{\alpha})^2} - \left(\Sigma\frac{1}{1+r^{\beta}}\right)^2 + \Sigma\frac{1}{(1+r^{\beta})^2}\right\}.$$

Proceeding as before it may be shown that

$$\left(\Sigma\frac{1}{1+r^{\alpha}}\right)^2 - \left(\Sigma\frac{1}{1+r^{\beta}}\right)^2 = \frac{p-1}{2}.2/p.(i\sqrt{p})h,$$

and $$\Sigma\frac{1}{(1+r^{\alpha})^2} - \Sigma\frac{1}{(1+r^{\beta})^2} = 2/p(i\sqrt{p})h;$$

therefore

$$\left(\frac{d^2Z}{dx^2}\right)_{x=-1} = \frac{2}{i\sqrt{p}}.2/p.\tfrac{1}{2}.2/p(i\sqrt{p})h\left\{\frac{p-1}{2}-1\right\} = \frac{(p-3)h}{2}.$$

6. It may be noted that if Y be expanded in powers of $(x-1)$, the coefficients of $(x-1)$ and $(x-1)^2$ are numerically pH times the coefficients of x and x^2, respectively, in Y, and that if Z be expanded in powers of $(x+1)$, the coefficients of $(x+1)$ and $(x+1)^2$ are numerically h times the coefficients of x and x^2 in Y.

7. The values, for $p = 37$ and 43, of Y and Z in the transformation $4\dfrac{x^p-1}{x-1} = Y^2 - (-1)^{\frac{1}{2}(p-1)}pZ^2$, have been calculated by the method given by Mathews, *Theory of Numbers*, Part I., p. 216. They are

$$Y_{37} = 2x^{18} + x^{17} + 10x^{16} - 4x^{15} + 15x^{14} - 5x^{13} + 17x^{12} - 8x^{11} + 11x^{10}$$
$$- 4x^9 + 11x^8 - 8x^7 + 17x^6 - 5x^5 + 15x^4 - 4x^3 + 10x^2 + x + 2,$$
$$Z_{37} = x^{17} + 2x^{15} - x^{14} + 3x^{13} - x^{13} + 2x^{11} - x^{10} + 2x^9 - x^8 + 2x^7 - x^6$$
$$+ 3x^5 - x^4 + 2x^3 + x,$$

$$Y_{43} = 2x^{21} + x^{20} - 10x^{19} + 6x^{18} + 16x^{17} - 20x^{16} - 4x^{15} + 27x^{14} - 15x^{13}$$
$$- 7x^{12} + 17x^{11} - 17x^{10} + 7x^{9} + 15x^{8} - 27x^{7} + 4x^{6} + 20x^{5} - 16x^{4}$$
$$- 6x^{3} + 10x^{2} - x - 2,$$

$$Z_{43} = x^{20} - 2x^{18} + 2x^{17} + 2x^{16} - 4x^{15} + x^{14} + 3x^{13} - 3x^{12} + x^{11} + x^{10}$$
$$- 3x^{9} + 3x^{8} + x^{7} - 4x^{6} + 2x^{5} + 2x^{4} - 2x^{3} + x.$$

Hence Legendre's rule for the calculation of Y first fails for $p = 43$. This fact might have been easily ascertained, and the true value of Y got, with a fair amount of probability, by checking Legendre's rule by means of the relation

$$\left(\frac{dY}{dx}\right)_{x=1} = 2\sqrt{p}\,(-1)^{\frac{1}{2}(h-1)}\,pH,$$

or also
$$\left(\frac{d^2Y}{dx^2}\right)_{x=1} = 2\sqrt{p}\,(-1)^{\frac{1}{2}(h-1)}\,p\,.\frac{p-3}{2}\,.H.$$

The values of Y, for primes of form $4n+3$, up to $p = 43$, have been expressed in powers of $x-1$. Writing X for $x-1$ they are

$$Y_7 = 2X^3 + 7(X^2 + X),$$

$$Y_{11} = 2X^5 + 11(X^4 + 2X^3 + 2X^2 + X).$$

$$Y_{19} = 2X^9 + 19(X^8 + 4X^7 + 9X^6 + 13X^5 + 13X^4 + 9X^3 + 4X^2 + X),$$

$$Y_{23} = 2X^{11} + 23(X^{10} + 5X^9 + 14X^8 + 23X^7 + 19X^6 - 3X^5 - 26X^4$$
$$- 29X^3 - 15X^2 - 3X),$$

$$Y_{31} = 2X^{15} + 31(X^{14} + 7X^{13} + 29X^{12} + 78X^{11} + 139X^{10} + 154X^9$$
$$+ 67X^8 - 95X^7 - 223X^6 - 236X^5 - 159X^4$$
$$- 72X^3 - 21X^2 - 3X),$$

$$Y_{43} = 2X^{21} + 43(X^{20} + 10X^{19} + 62X^{18} + 268X^{17} + 861X^{16}$$
$$+ 2140X^{15} + 4229X^{14} + 6777X^{13} + 8939X^{12} + 9821X^{11}$$
$$+ 9079X^{10} + 7127X^9 + 4789X^8 + 2769X^7 + 1376X^6 + 580X^5$$
$$+ 201X^4 + 54X^3 + 10X^2 + X).$$

This seems to suggest that not only the last two but all the last $\frac{p-3}{4}$ terms in the bracket have the sign of $2\sqrt{p}(-1)^{\frac{1}{2}(h-1)}$, that is, of $(-1)^a$, where a is the number of residues less than $\frac{p}{2}$.

Shrewsbury School.

ON SOME GEOMETRICAL DISSECTIONS.

By *H. M. Taylor*, M.A., F.R.S.

§ 1. THERE are in the elements of Euclid many propositions which deal with the equality of the areas of two rectilinear figures. The fundamental principle on which Euclid treated this subject is that of superposition, which, as early as in the 4th Proposition of the first book, is made use of to prove the equality of two triangles.

It is perhaps scarcely too much to say that most of the propositions with reference to the equality of two areas depend on the axiom or definition that if equal figures be added to or subtracted from equal figures, the sums or the remainder (as the case may be) are figures of equal area. This principle is made use of in Proposition 35 to prove that parallelograms of equal altitude on the same base are equal in area, and again in Proposition 43 to prove that the complements of parallelograms about a diagonal of a parallelogram are equal in area.

If we except some of the propositions of the second book, Euclid seems not to have proved the equality of two areas by showing how to divide them into the same number of parts which would be capable of being exactly superposed in pairs.

It seems likely that an account of some simple cases where the equality of two areas can be proved by this direct method may be of interest to those engaged in teaching geometry.

§ 2. Most of the dissections which are dealt with in this paper have been cut out in cardboard, and the problem of fitting the pieces together so as to form the two corresponding figures of equal area has proved, even to persons little acquainted with geometry, a fascinating pastime.

It has been somewhat difficult to decide on the order in which the different dissections should be arranged. It will be seen that the transformation of a figure into another of given shape by division into a definite number of parts generally requires some condition; and that if this condition be not satisfied, the corresponding dissection requires division

into a larger number of parts. In some of the simpler dissections the number of parts into which a given figure must be divided is discussed; but in some of the more complicated dissections it has been thought sufficient to give two or three cases of the problem under consideration without discussing the general problem.

§ 3. It may be observed here that in each of the figures which are drawn in connection with this paper there are two diagrams, one of which represents the geometrical area as it is supposed to be given, and the other the area into which it is transformed.

In the one diagram the angular points of the given area, and the points which are used in the construction for the transformation, are represented by unaccented letters. In the other diagram unaccented letters represent points which remain unchanged in the transformation, while accented letters represent the changed positions of the points in the first diagram which are represented by the same letters unaccented.

§ 4. We will begin by considering the case of two parallelograms of equal bases and equal altitudes.

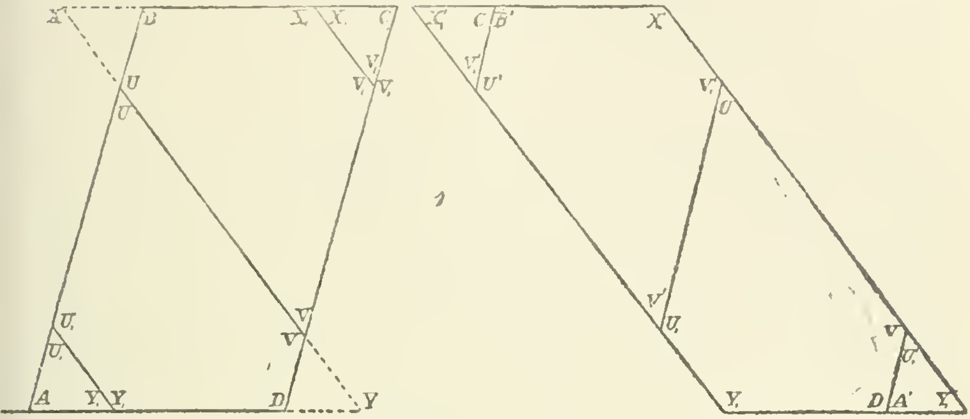
Transformation of a parallelogram into another of equal base.

Case I. Let $ABCD$ be a parallelogram. Take two points X, Y in BC, AD respectively, such that the line XY is equal to a side of another parallelogram of the same base and altitude as $ABCD$. Then, if the line XY be cut, and one of the two figures $ABXY$, $YXCD$ be translated, so that the lines AB, DC coincide, we shall have a parallelogram of the required shape.

This dissection is so simple that a figure has not been considered necessary. It holds good as long as the points X, Y fall within the sides BC, AD. The limiting positions of XY are the diagonals AC, BD. If the line to which XY is to be equal is of such length that both the points X, Y can lie within the lines BC, AD respectively, then the actual position of the line XY admits of some variation: it might be drawn through the centre O of the parallelogram, or on either side of this central position between it and a parallel position passing through one or other of the extremities of a diagonal.

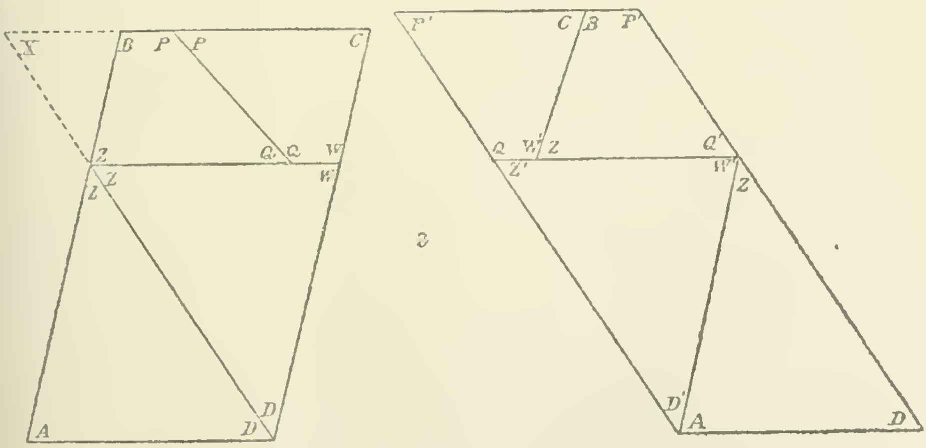
§ 5. *Case II.* Let us now assume that the line XY is drawn through the centre O, and that it is of such a length

that the points XY lie in the lines CB, AD produced, and let XB be less than BC (fig. 1).



Let XY meet AB, CD in U, V: take in AB, CD two points U_1, V_1, such that AU_1 and CV_1 each equal BU, and draw U_1Y_1, V_1X_1 parallel to XY, meeting AD, BC in Y_1, X_1 respectively. Then, if the lines UV, U_1Y_1, X_1V_1 be cut, the figures AU_1Y_1, U_1UVDY_1, UBX_1V_1V, X_1CV_1 can be fitted together so as to form the required parallelogram. Here we have a dissection into four parts.

§ 6. *Case III.* Let us next assume that the line XY is so drawn that Y coincides with D (fig. 2), and that the point X



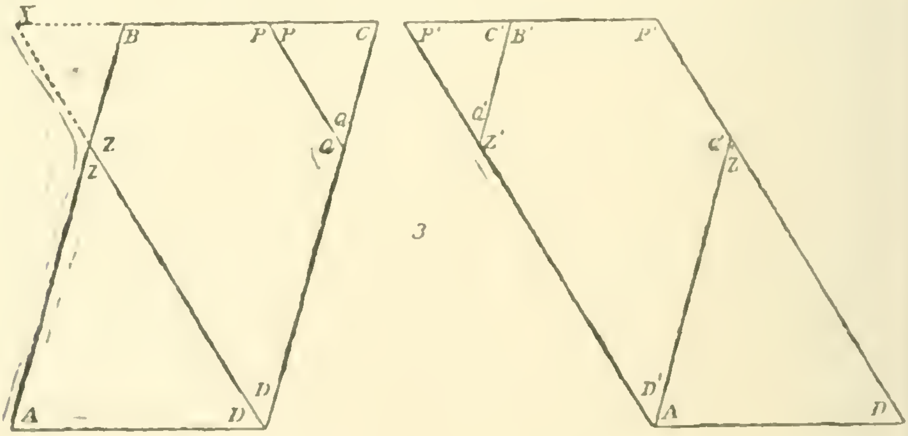
lies in CB produced. Further, assume that XB is less than BC, and let XD meet AB in Z. Draw ZW parallel to AD to meet CD in W. Through the centre of the parallelogram

$ZBCW$ draw the line PQ parallel to XD to meet BC, ZW in P, Q. Then, if the lines ZD, ZW, PQ be cut, the four figures AZD, ZWD, $ZBPQ$, $PCWQ$ can be fitted together so as to from the required parallelogram. Here again we have a dissection into four parts.

§ 7. This last method of dissection can be readily adapted to the case of any pair of parallelograms of equal bases.

If the line DX meets the line AB in Z, and ZB is greater than m times and less than $(m+1)$ times AZ, where m is an integer, measure off from ZB, ZZ_1, Z_1Z_2, Z_2Z_3, ..., $Z_{m-1}Z_m$, each equal to AZ. Draw ZW, Z_1W_1, ..., Z_mW_m parallel to AD to meet CD in W, W_1, ..., W_m; draw the diagonals WZ_1, W_1Z_2, ..., $W_{m-1}Z_m$, and draw PQ through the centre of the parallelogram Z_mBCW_m parallel to XD to meet BC, Z_mW_m in P, Q; then, if the lines DZ, ZW, WZ_1, Z_1W_1, W_1Z_2, ..., $W_{m-1}Z_m$, Z_mW_m, PQ be cut, the figures thereby formed can be fitted together so as to form the required parallelogram.

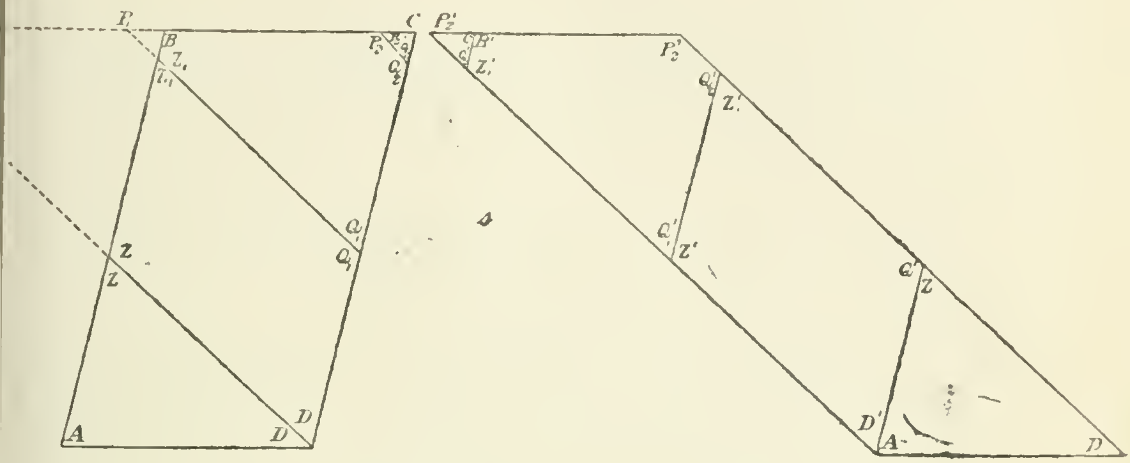
§ 8. *Case IV.* Next let us assume that the point Y is taken to coincide with D, and that the point X lies in CB produced, and that XB is less than BC (fig. 3).



Let DX meet AB in Z: drawn a line PQ cutting BC, CD in P, Q, such that the triangle PCQ is equal to the triangle XBZ. Then if the lines ZD, PQ be cut, the figures AZD, $ZBPQD$, PCQ can be fitted together so as to form the required parallelogram.

Here we have a dissection into three parts.

§ 9. *Case V.* (Fig. 4). Next let the line XB be greater



than BC, and less than twice BC. Take XP_1, P_1P_2 each equal to BC, and draw P_1Q_1, P_2Q_2 parallel to XD to meet CD in Q_1, Q_2, and let P_1Q_1 meet AB in Z_1. Then if the lines ZD, Z_1Q_1, P_2Q_2 be cut, the figures AZD, ZZ_1Q_1D, $Z_1BP_2Q_2Q_1$, P_2CQ_2 can be fitted together so as to form the required parallelogram.

Here we have a dissection into four parts.

§ 10. If, when the line DX is drawn equal to one of the sides of the required parallelogram, the line XB is greater than m times and less than $(m+1)$ times BC, where m is an integer, we measure off from XC lines XP_1, P_1P_2, ..., P_mP_{m+1}, each equal to BC. Draw lines P_1Q_1, P_2Q_2, ..., $P_{m+1}Q_{m+1}$ parallel to XD to meet AB in Z_1... and CD in Q_1..., then, if the lines ZD, Z_1Q_1, Z_2Q_2, ..., Z_mQ_m, $P_{m+1}Q_{m+1}$ be cut, the figures

$$AZD,\ ZZ_1Q_1D,\ Z_1Z_2Q_2Q_1,\ \ldots,\ Z_mBP_{m+1}Q_{m+1}Q_m,\ P_{m+1}CQ_{m+1}$$

can be fitted together so as to form the required parallelogram.

Here we have a dissection into $(m+3)$ parts.

Transformation of a parallelogram into an isogonal parallelogram.

§ 11. Let $ABCD$, $EFGH$ be two isogonal parallelograms, and let them be placed so that E coincides with C, and BCH, FCD are straight lines.

Draw the line XCY parallel to BF to meet AB, FG in X, Y. And draw the line AG cutting CD, CH in U, V.

Then the triangles XBC, CFY, UDA, GHV are equal in all respects.

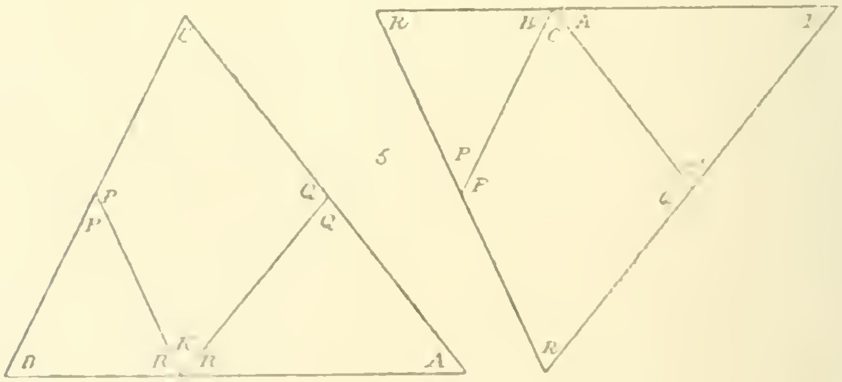
We have therefore only to divide the two parallelograms $XCUA$, $CYGV$ into parts which will fit each other in pairs. This may be done by any of the methods explained above.

Transformation of a triangle into another triangle having two sides the same.

§ 12. Let ABC be a triangle, and R the middle point of the side AB. Draw the line CR, and let it be cut. Then, if the triangle CRA be rotated through two right angles round the point R, we obtain a triangle having two sides equal to two sides of the triangle ABC.

Transformation of a triangle into a triangle having one side the same and one side equal to a given length.

§ 13. (Fig. 5). Let ABC be a triangle and P, Q the



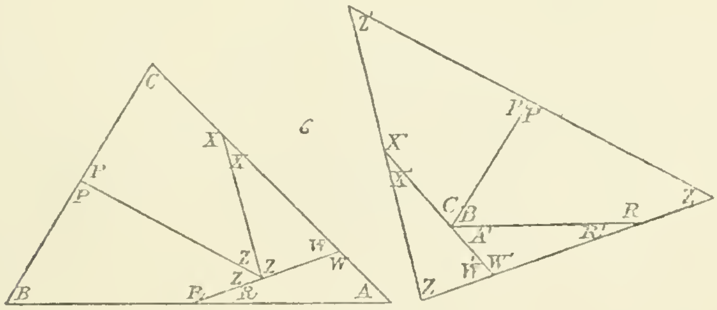
middle points of BC, CA, and R any point in AB, then if the lines PR, QR be cut, and the triangles PBR, QAR be rotated through two right angles round the points P, Q, we obtain a triangle having R for a vertex. In this dissection the new triangle has the side opposite R equal to the side AB of the triangle ABC, but the point R may be chosen arbitrarily so that either of the other sides or any of the angles of the triangle may, within certain limits, have given values.

Transformation of a triangle into another of given shape.

§ 14. *Method I.* Let ABC be a triangle and P, Q the middle points of BC, CA. Take a point R in AB such that RP is equal to half one of the sides of the required triangle, and cut the lines RP, RQ, and rotate the triangles PBR, QAR through two right angles round P, Q. This process gives a new triangle RST, having the side RS equal to one of the sides of the required triangle.

Next, by a second application of this method, take V the middle point of ST, and take a point L in RS such that LQ is equal to half of a second side of the required triangle. Then cut the lines LQ, LV and rotate the triangles QLR, VLS through two right angles round Q, V, then we obtain a triangle LMN having two sides equal to two sides of the required triangle; and, as the triangles are of equal area, LMN is of the required shape.

§ 15. *Method II.* (Fig. 6). We have just given a method

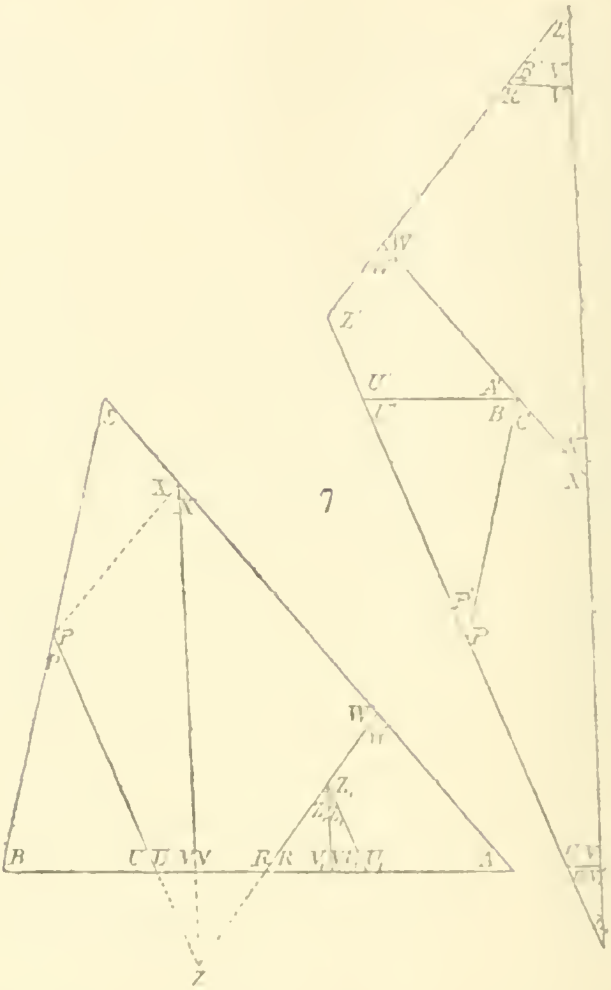


of transforming a triangle into one of given shape by a dissection into six parts. There is, however, a simpler method of performing this transformation.

Let ABC be the triangle, and let P, Q, R be the middle points of BC, CA, AB. Take a point X in CA such that PX is equal to half one of the sides of the required triangle, and let CX be less than CQ. On PX, on the side remote from C, construct a triangle PXZ having each of its sides equal to half of one of the sides of the required triangle. Draw the line RZ (which is parallel to PX), and let it be produced to meet CA in W. Then XW will be equal to CQ.

Cut the lines RZW, ZP, ZX. Then, if the figures $PCXZ$, RAW be rotated through two right angles round P, R, and the triangle XZW be translated so that the two new points X' and the two new points W' coincide, we shall have a triangle of the required shape.

§ 16. *Method III.* (Fig. 7). We will now take a case of



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the transformation of one triangle into another of given shape when the construction just given fails.

Let ABC be a triangle, and P, Q, R the middle points of the sides BC, CA, AB. Construct the triangle PXZ, having its sides equal to half the sides of the required triangle, so that the point X lies in CA. In this case let the point X lie in CQ and the point Z outside the triangle ABC, and let ZR, which is parallel to PX, be drawn and produced to meet CA in W.

Draw PZ, XZ, and let them meet AB in U, V. Take a point Z_1, such that R is the middle point of $Z\,Z_1$, and draw Z_1U_1, Z_1V_1 parallel to ZU, ZV to meet AB in U_1, V_1. Cut the lines PU, XV, RW, Z_1U_1, Z_1V_1. Then, if the figures PBU, $XVRW$, $Z_1U_1V_1$ be rotated through two right angles round P, X, R, and the figures RZ_1V_1, WZ_1U_1A be translated

so that the point V_1 coincides with the new position of V and A with C, we shall have a triangle of the required shape.

§ 17. Next let us consider the case of the dissection of two parallelograms which are merely equal in area.

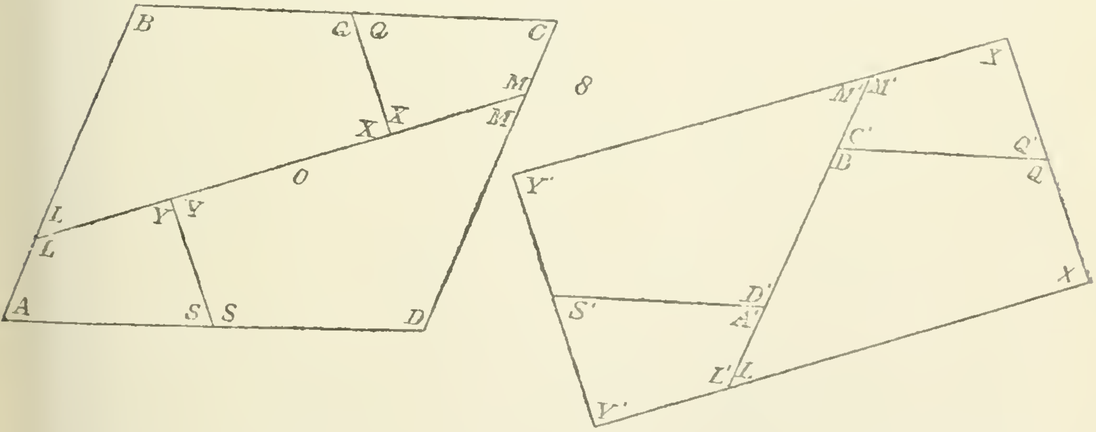
Let $ABCD$, $EFGH$ be two such parallelograms. It will be possible to place the two parallelograms so that the lines joining corresponding vertices, say AE, BF, CG, DH, are parallel. Then, if BF meet AD in X and EH in Y, and if DH meet BC in U and FG in V (these points being in each case internal), the lines BX, FY, UD, VH are all equal.

We have therefore only to divide two pairs of triangles and one pair of parallelograms into parts which will fit each other in pairs, the triangles and the parallelograms being respectively of equal bases and altitudes.

There are, however, other simpler methods of dissection by which two parallelograms of equal area which can be transformed into one another. We will now proceed to consider such transformations.

Transformation of a parallelogram into a parallelogram of given shape.

§ 18. *Method I.* (Fig. 8). Let $ABCD$ be a parallelogram



and P, Q, R, S the middle points of AB, BC, CD, DA. Draw through O, the centre of the parallelogram $ABCD$, the line LOM meeting AB, CD in L, M. Draw two parallel lines QX, SY meeting LOM in X, Y, and making the angles at X, Y equal to the angles of the required parallelogram. Let the ratio of QX to OM be equal to the ratio of the sides of the required parallelogram. Then, if the figures $ALYS$, $CQXM$ be rotated through two right angles round L, Q, and

the figure $DMYS$ be translated so that the point D coincides with the new position of A, we obtain a parallelogram of the required shape.

Let a, b, α be the sides and an angle of the given parallelogram, and a', b', α' the corresponding quantities for the new parallelogram, then $ab \sin\alpha = a'b' \sin\alpha'$, and if the angle QOM be θ we must have

$$a' = a\frac{\sin\theta}{\sin\alpha}, \quad \text{and} \quad b' = b\frac{\sin\alpha}{\sin\theta}.$$

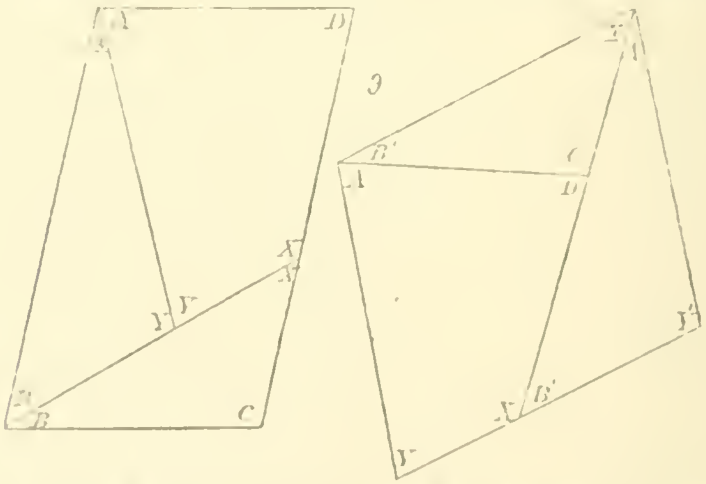
Therefore θ must be found from an equation of the form

$$\sin\theta = \frac{b}{b'}\sin\alpha.$$

An interesting special case of this dissection is when the new parallelogram is a square.

Transformation of a parallelogram into a parallelogram of given shape.

§ 19. *Method II.* (Fig. 9). Let $ABCD$ be a parallelogram.



Take a point X in CD, such that BX is equal to one of the sides of the required parallelogram; and take a point Y in BX, such that AY is equal to one of the adjacent sides of the required parallelogram.

Let BX, AY be cut. Then, if the figures AYB, BXC be translated so that B of the first figure coincides with X of the unmoved figure, and BC of the second figure coincides with AD of the unmoved figure, we obtain a parallelogram of the required shape.

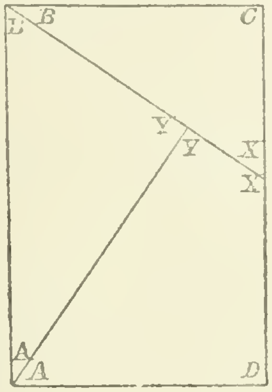
In this form of a dissection we obtain two solutions if there are two positions of Y between B and X. If either of the

positions of Y lies in the line BX produced, either beyond B or beyond X, the corresponding dissection requires division into a greater number of parts than three.

It will be observed that in the two solutions which correspond to the two positions of Y, the parallelograms obtained by the construction given above are not superposable unless one of them be turned upside down.

Transformation of a rectangle into another of given shape.

§ 20. (Figs. 10 and 11). Let $ABCD$ be the given rect-



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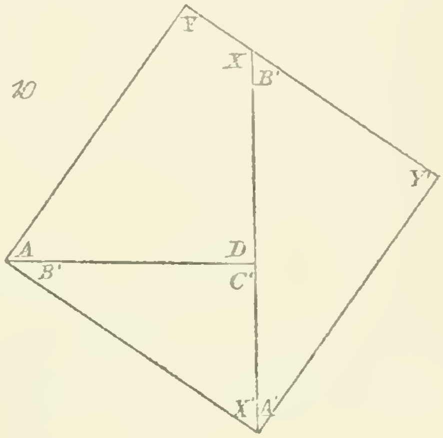

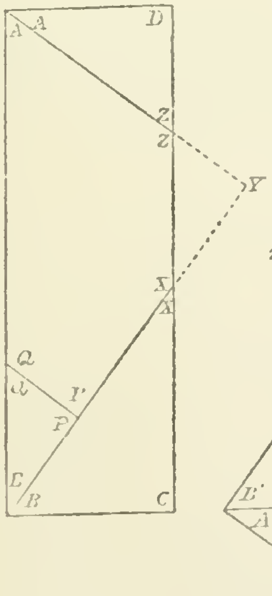

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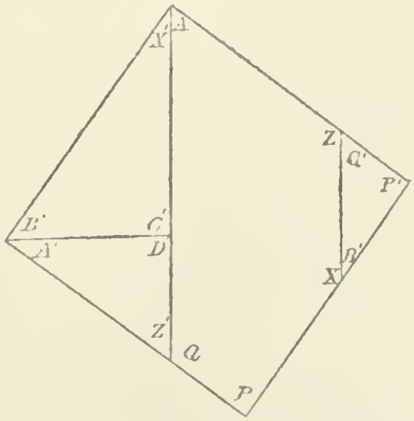


angle. Take a point X in CD, such that BX is equal to one of the sides of the required rectangle. Draw AY perpendicular

to BX. Then, if the point Y falls within the line BX, the three parts ABY, $AYXD$, BCX can be fitted together so as to form the required rectangle; but if the point Y falls without the line BX, the number of parts into which the rectangle $ABCD$ must be divided so as to form the required rectangle is increased.

It will perhaps be considered sufficient to discuss the different cases which may occur when the required rectangle is a perfect square.

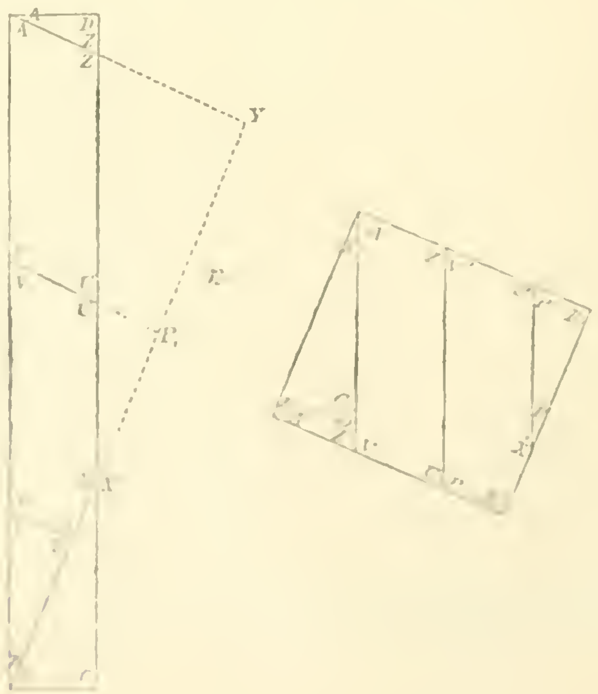
Let $ABCD$ be the given rectangle, and let the side AB be n, where n is a quantity greater than unity, and the side BC be 1. Take a point X in CD, such that $BX = \sqrt{n}$, and draw AY perpendicular to BX. Then $AY = \sqrt{n}$ and $BY = \sqrt{(n^2 - n)}$.

Case I. (Fig. 10). If $\sqrt{(n^2 - n)} < \sqrt{n}$, or $n < 2$, the point Y will fall within the line BX, and the required dissection requires only three parts.

Case II. (Fig. 11). If $\sqrt{(n^2 - n)} > \sqrt{n}$ but $< 2\sqrt{n}$, or $n > 2$ but < 5, then the dissection requires four parts.

In this case the dissection is completed by measuring off from BX, BP equal to XY, and drawing PQ parallel to AY to meet AB in Q. If AY cut CD in Z, then the parts required are $AQPXZ$, AZD, QBP, BCX.

The general case. If $\sqrt{(n^2 - n)} > m\sqrt{n}$ but $< (m+1)\sqrt{n}$, where m is an integer, or $n > m^2 + 1$ but $< m^2 + 2m + 3$, the dissection requires $m + 3$ parts (see fig. 12, where $n = 7$, $m = 2$).



In this case the dissection is completed by, from YB, measuring off YP_1, P_1R, each equal to XB, and drawing P_1V, RP parallel to AY to meet AB in V, P.

Transformation of a triangle into a parallelogram.

§ 21. Let ABC be a triangle and P, Q be the middle points of BC, CA. If the line PQ be cut and the triangle CPQ be rotated through two right angles round P or Q, we shall obtain a parallelogram. In this dissection the parallelogram has a base and one angle in common with the triangle.

Transformation of a triangle into a parallelogram having the same base as the triangle and given angles.

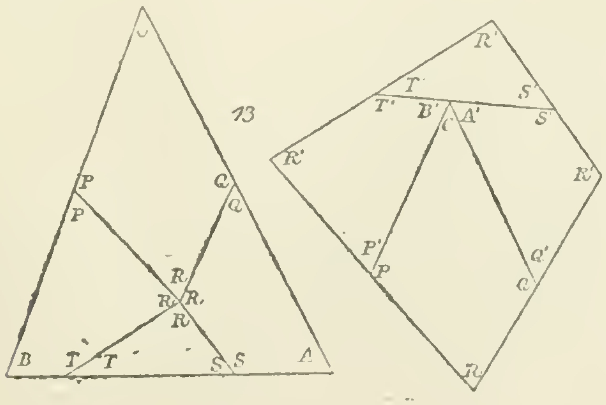
§ 22. Let ABC be a triangle and P, Q the middle points of BC, CA. If any point R be taken in PQ and the lines PQ, CR be cut, then, if the triangles PCR, QCR be rotated through two right angles round the points P, Q, we obtain a parallelogram. In this dissection the new parallelogram has a base in common with the triangle, but its angles are capable of being chosen arbitrarily within certain limits.

Next, let R be taken beyond Q, and AS be drawn parallel to CR to meet PQ between P and Q; then, if the lines PQ, AS be cut, the triangle ASQ may be rotated through two right angles round Q into the position CQR, and then the whole triangle CPR rotated through two right angles round P. In this dissection the new parallelogram has a definite base, the same as the triangle, but its angles, within certain limits, are arbitrary.

If the line AS cut PQ beyond P, then the number of parts into which the triangle ABC must be divided will be increased.

Transformation of a triangle into a quadrilateral.

§ 23. (Fig. 13). Let ABC be a triangle and P, Q the

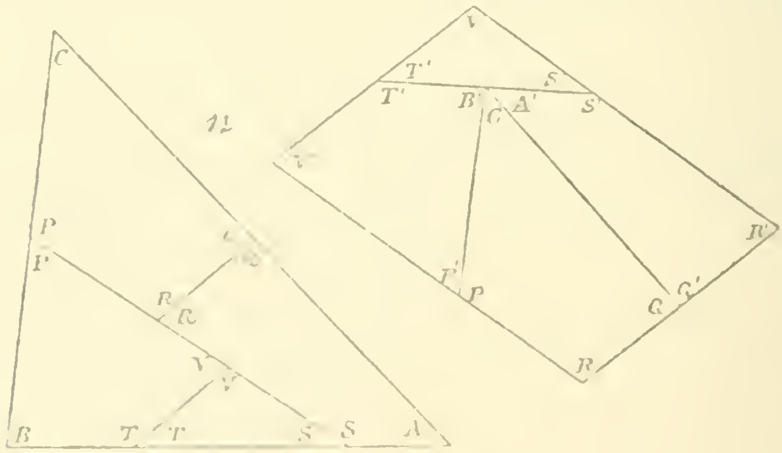


middle points of BC, CA. Let a point R be taken within the figure $PQAB$, and let two points S, T be taken in AB, such that ST is equal to PQ (S being the point nearer A). Let the lines RP, RQ, RS, RT be cut. Then, if the figures $PRTB$, $QRSA$ be rotated through two right angles round P, Q, and the figure RST be translated so that the line ST may coincide with the line made up of the new positions of SA and BT, we shall obtain a quadrilateral whose sides will be the doubles of the lines RP, RQ, RS, RT, and whose angles are equal to the angles at R.

It will be noticed that in this general case the four points which form the vertices of the new quadrilateral met at the same point R in the original triangle. If the quadrilateral is to be a trapezium or a parallelogram, this is not necessary, as will be seen in the next case.

Transformation of a triangle into a parallelogram of given shape.

§ 24. (Fig. 14). Let ABC be a triangle and P, Q the



middle points of BC, CA. On PQ, on the side remote from C, construct a triangle PQR, having the side QR equal to half of one of the sides, and the angle PRQ equal to one of the angles of the required parallelogram.

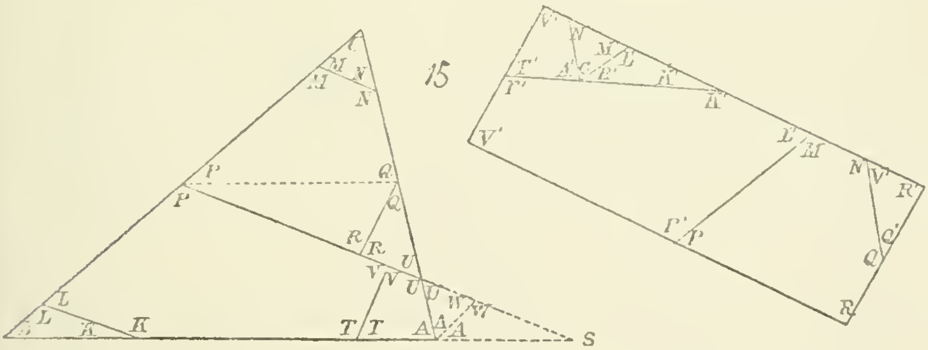
Produce PR to meet AB in S, and from SB cut off ST equal to PQ. Draw TV parallel to RQ to meet PS in V. Let the lines PRS, QR, and TV be cut. Then, if the figures $PVTB$, $QRSA$ be rotated through two right angles round P, Q, and the triangle STV be translated so that TS coincides with the line made up of the new positions of SA and BT, we obtain a parallelogram of the required shape.

It will be observed that the triangles PQR, STV are equal in all respects, and that the line PS is equal to the second side of the required parallelogram.

§ 25. A special case of this dissection of a triangle has lately attracted some attention in the columns of one of the daily newspapers. The problem, "To divide an equilateral triangle into four parts which could be fitted together so as to form a perfect square," was proposed in the *Daily Mail* of February 1st, 1905, and the solution was given on February 8th by Mr. H. E. Dudeney, the proposer of the question.

In this special case, if the side of the equilateral triangle ABC be 2, the length of the line PS is equal to the fourth root of 3. The angles at R and V are right angles.

§ 26. (Fig. 15). The construction given above for the



dissection of a triangle into a parallelogram of given shape fails unless the point R fall within the figure $PQAB$, and the point S within the line AB, and unless the line SB be not less than PQ.

We will now take the case where the line PR produced meets AC in U, and BA produced in S and AS is less than PQ.

Take a point T in AB, such that ST is equal to PQ, and draw AW parallel to BC and TV parallel to RQ to meet PS in W, V. Take points K, L, M, N in AB, BC, BC, CA respectively, so that $BK = AS$, $BL = AW$, $CM = AW$, and $CN = AU$, and draw KL, MN. Then cut the lines PU, QR, TV, KL, MN. First shift the triangles KBL, MCN into the positions SAW, WAU. Then, if the figures $PVTKL$, QRU be rotated through two right angles round P, Q, and the whole triangle TVS be translated so that ST coincides with the new position of KT, we have a parallelogram of the required shape.

Transformation of a parallelogram into a triangle.

§ 27. The next problem which we will discuss is to divide a parallelogram into parts which will fit together and form a triangle.

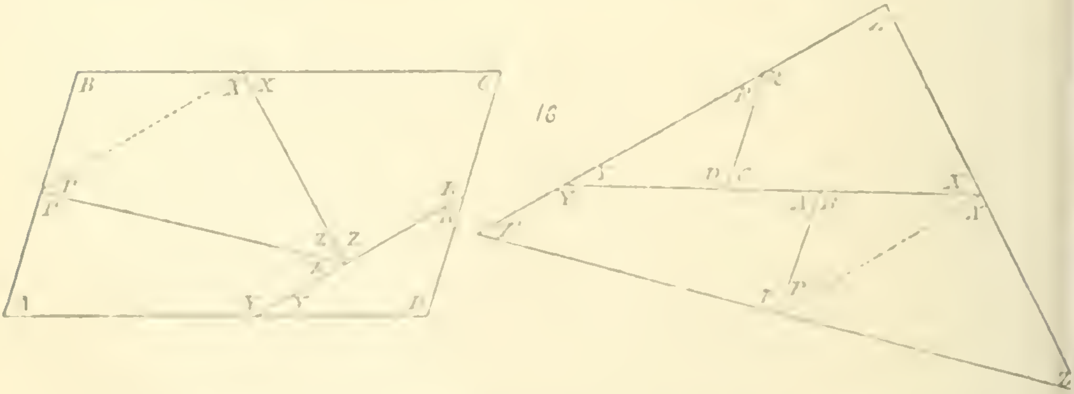
Let $ABCD$ be a parallelogram, and let P be the middle point of the side AB. Draw the line PC, and let this line be cut. Then, if the triangle PBC be rotated through two right angles round P, we shall obtain a triangle.

Here we have a dissection into two parts. The triangle obtained by this dissection has one side and one angle the same as the given parallelogram.

§ 28. Next, let $ABCD$ be a parallelogram, P, R the middle points of the sides AB, CD, and X a point in the side BC. Draw the lines PX, XR, and let these lines be cut. Then, if the triangles PBX, XCR be rotated through two right angles round P, R, we shall obtain a triangle. Here we have a dissection into three parts: but the triangle obtained by this dissection has an altitude equal to that of the given parallelogram, while its angles are capable of considerable variation.

Transformation of a parallelogram into a triangle of given shape.

§ 29. (Fig. 16). Since the new triangle is of given shape and area, the sides must be determinate.

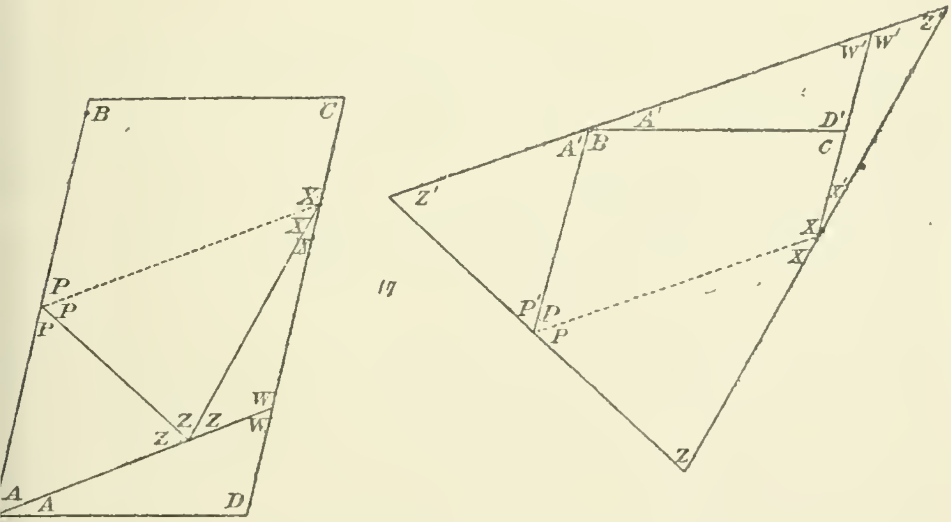

16

Let $ABCD$ be a parallelogram and P, R the middle points of AB, CD. Take a point X in BC, such that PX is equal to half one of the sides of the required triangle, and draw RY

parallel to PX to meet DA in Y. Take a point Z in RY, such that XZ is equal to half of one of the other sides of the required triangle, and draw PZ. Cut the lines PZ, XZ, YZR. Then, if the figures $PZYA$, $XZRC$ be rotated through two right angles round P, X, and the triangle RYD be translated so that D coincides with the new position of C, we obtain a triangle of the required shape.

Perhaps one of the most interesting special cases of this transformation is to cut up a square into four parts which will form an equilateral triangle. In this special case if AB, the side of the square, be equal to the square root of 3, then the line PX is equal to 1.

§ 30. (Fig. 17). Next, let us consider the case when the

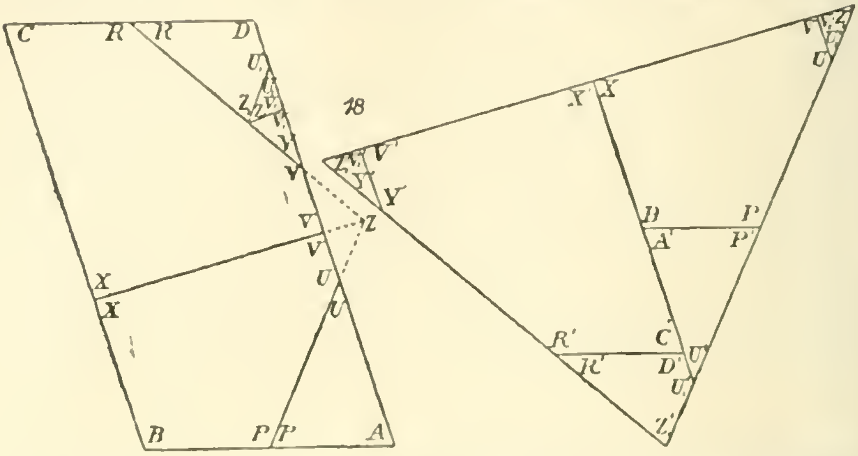


17

point X is taken in the side CD of the parallelogram so that the line PX is equal to half of one of the sides of the required triangle, and the point Z is within the parallelogram. Draw AZ, which is parallel to PX, and let it be produced to meet CD in W. Draw ZP, ZX. Cut the lines ZP, ZX, AW. Then, if the figures PZA, XZW be rotated through two right angles round the points P, X, and the figure AWD be translated so that the point A coincides with the point B, we shall have a triangle of the required shape.

The construction here given for the transformation of a given parallelogram fails unless the point X lies in the side CD, and the point Z lies within the parallelogram. When either of these conditions is not satisfied the number of parts into which the parallelogram must be divided is increased.

§ 31. (Fig. 18). We will next consider the case when the



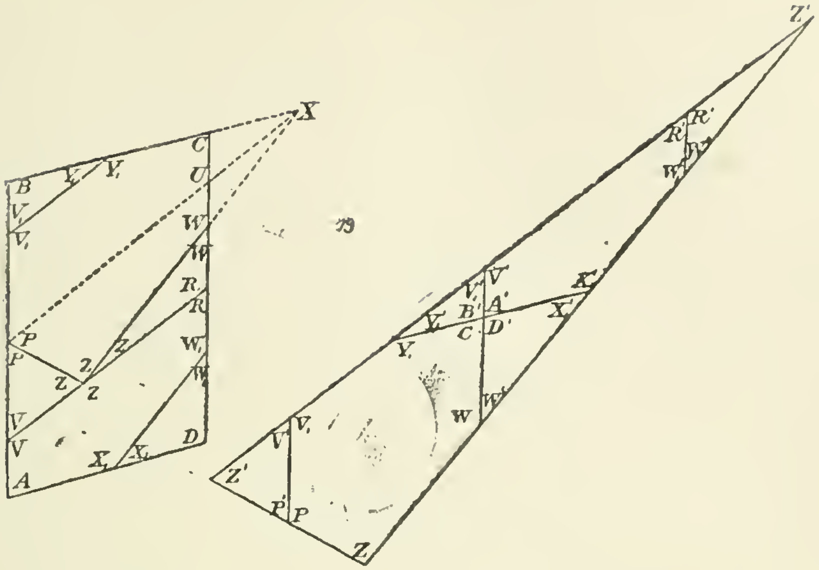
point X lies in the side BC, but the point Z lies outside the parallelogram beyond the side AD.

Let $ABCD$ be a parallelogram, and let P, R be the middle points of AB, CD. Draw the two parallel lines PX, RY so that PX is equal to half one of the sides of the required triangle, X lying in BC and Y in AD, and take a point Z in RY produced beyond Y, such that the triangle PXZ is similar to the required triangle.

Draw the lines ZP, ZX, meeting AD in U, V. Take points Z_1, U_1, V_1, such that Y is the middle point of the pairs of points Z, Z_1; U, U_1; V, V_1, and draw Z_1U_1, Z_1V_1. Then cut the lines PU, XV, RY, Z_1U_1, and Z_1V_1. Let the figures PAU, $XVYRC$ be rotated through two right angles round the points P, X, and let the figure RZ_1U_1D be translated so that RD may coincide with the new position of RC. Then if the triangle $Z_1U_1V_1$ be shifted on to the position ZUV, and the triangle Z_1YV_1 be translated so that Y and V_1 coincide with the new positions of Y, V, we shall have the triangle of the required shape.

§ 32. (Fig. 19). We next consider the case when X lies in BC produced.

Let $ABCD$ be a parallelogram, and let P, R be the middle points of AB, CD. Take a point X in BC produced, such that PX is equal to half of one of the sides of the required triangle, and let PX meet CD in U. Construct on PX on the side remote from B the triangle PXZ, having its sides half the sides of the required triangle. Draw RZ and let it be produced to meet AB, and DA produced in V, Y. Let ZX meet CD in W. Take points V_1, W_1, such that P is the middle point of VV_1, and R is the middle

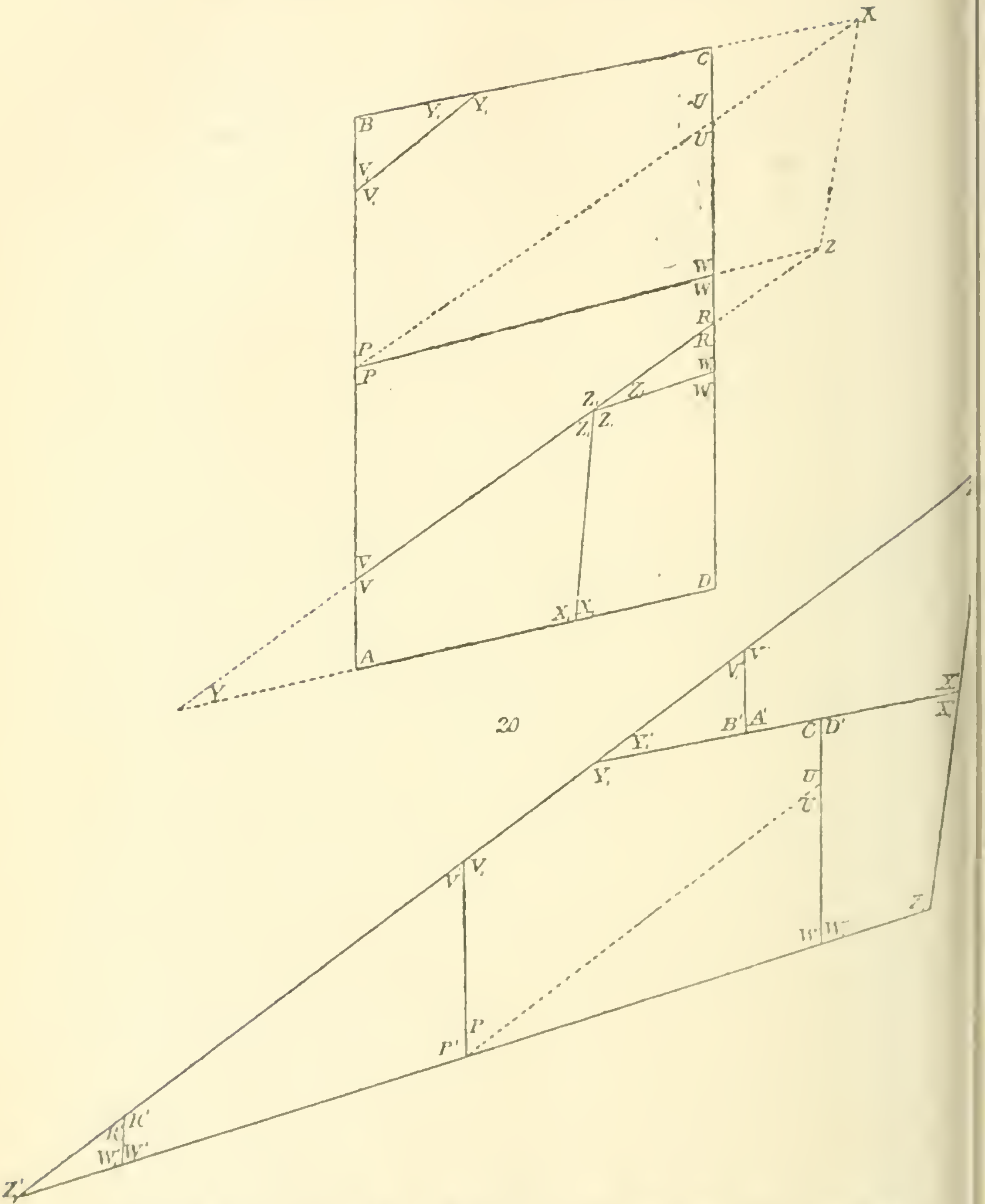


point of WW_1. Draw V_1Y_1 parallel to PX to meet BC in Y_1, and W_1X_1 parallel to WX to meet AD in X_1. Cut the lines VZR, ZP, ZW, V_1Y_1, X_1W_1. Then, if the figures PYZ, Y_1BV_1, W_1DX_1, ZWR be rotated through two right angles round the points P, Y_1, R, R, and then the figure $AVRW_1X_1$ be translated so that A coincides with the new position of B, we shall have a triangle of the required shape.

§ 33. (Fig. 20). Next, we will consider a case of transformation of a parallelogram into a triangle where the ordinary construction given above fails in two respects.

Let $ABCD$ be a parallelogram, and P, R be the middle points of AB, CD. Let a point X be taken in BC produced such that PX is equal to half one of the sides of the required triangle (CX being less than BC), and let on PX be constructed a triangle PXZ having its sides equal to half the sides of the required triangle (the point Z falling outside the parallelogram). PZ is not necessarily parallel to BX.

Let PX meet CD in U. Draw PZ and let it meet CD in W. Draw ZR and let it be produced to meet AB and DA produced in V, Y. Take points V_1, Z_1, W_1, such that P is the middle point of V, V_1, and R the middle point of the pairs Z, Z_1, W, W_1. Draw V_1Y_1 parallel to PX to meet BC in Y_1, and Z_1X_1 parallel to ZX to meet AD in X_1, and join Z_1W_1. Cut the lines VR, PW, V_1Y_1, Z_1W_1, and X_1Z_1. Then, if the figures $PVRW$, $W_1Z_1X_1D$, Y_1BV_1 be rotated through two right angles round the points P, R, Y_1, and the figures VAX_1Z_1, Z_1RW_1 be translated so that the points A, W_1



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coincide with the new positions of B, W, we shall have a triangle of the required shape.

Transformation of a trapezium into a triangle of given shape.

§ 34. Let $ABCD$ be a trapezium, of whose sides AD, BC are parallel, and AB, CD are not parallel. Let P, R be the middle points of AB, CD.

Take a point X in BC such that PX is equal to half of one of the sides of the required triangle. Construct on PX on the side remote from B the triangle PXZ, having its sides each equal to half of one of the sides of the required triangle.

Draw the line RZ, and let it be produced to meet AD in Y (YZR is parallel to PX).

Draw the lines ZP, ZX, and let the lines YZR, ZP, ZX be cut. Then, if the figures $PAYZ$, XZR be rotated through two right angles round P, X, and the triangle YDR translated so that the point D coincides with the new position of C, we shall have a triangle of the required shape.

Transformation of a quadrilateral into a triangle of base equal to a diagonal of the quadrilateral.

§ 35. Let $ABCD$ be a quadrilateral, and P, Q, R, S be the middle points of AB, BC, CD, DA. Draw PQ, RS, and let Z be any point in RS. Then, if RZS, PZ, QZ be cut, and the figures $PZSA$, $QZRC$ be rotated through two right angles round P, Q, and the triangle RSD be translated so that the point D coincides with B, we obtain a triangle having the side opposite Z equal to AC.

Transformation of a quadrilateral into a quadrilateral whose sides and angles satisfy two relations.

§ 36. Let $ABCD$ be a quadrilateral, and let P, Q, R, S be the middle points of AB, BC, CD, DA. Take a point O within the figure $PQCRSA$, such that twice the lines OP, OQ, OR, OS, and the angles POR, QOR, ROS, SOP, satisfy the given relations.

(The doubles of the lines OP, OQ, OR, OS are the sides, and the angles at O the angles of the required quadrilateral.)

Then, if the lines OP, OQ, OR, OS be cut, and the figures $POSA$, $QORC$ be rotated through two right angles round P, Q, and the figure $ORDS$ be translated so that the point D coincides with the point B, we obtain a quadrilateral satisfying the given relations. If the point O be the centre of the parallelogram $PQRS$, then the new quadrilateral will be a parallelogram.

ON VARIOUS EXPRESSIONS FOR h, THE NUMBER OF PROPERLY PRIMITIVE CLASSES FOR A DETERMINANT $-p$, WHERE p IS OF THE FORM $4n+3$, AND IS A PRIME OR THE PRODUCT OF DIFFERENT PRIMES.

(Second Paper.)

By *H. Holden.*

1. In a previous paper in the present volume (pp. 73–80) it was shown that

$$\Sigma \frac{1}{1-r^{a}} - \Sigma \frac{1}{1-r^{\beta}} = i(\sqrt{p})\, H$$

where
$$H = \frac{h}{2-2/p}.$$

Let q be an integer prime to p.

Then if
$$p \equiv l_1 \pmod{q},$$
$$2p \equiv l_2 \pmod{q}.$$
$$\vdots$$
$$(q-1)\,p \equiv l_{q-1} \pmod{q},$$

the residues $l_1, l_2, \ldots, l_{q-1}$ are incongruent (mod. q), and hence so are $q-l_1$, $q-l_2$, $\ldots$, $q-l_{q-1}$, and therefore they are $1, 2, 3, \ldots, (q-1)$ in some order.

If $q/p = 1$,

$$\Sigma \frac{1}{1-r^{a}} = \Sigma \frac{1}{1-r^{qa}} = \Sigma \frac{1+r^{a}+r^{2a}+\ldots+r^{(q-1)a}}{1-r^{qa}}.$$

Therefore
$$\Sigma \frac{r^{a}+r^{2a}+\ldots+r^{(q-1)a}}{1-r^{qa}} = 0.$$

Now
$$(q-1)\,\Sigma \frac{1}{1-r^{qa}}$$

$$\begin{aligned} = \quad & 1 + r^{qa} + \ldots + r^{(p-l_1)a} + \frac{r^{(p+q-l_1)a}}{1-r^{qa}} \\ & + 1 + r^{qa} + \ldots + r^{(2p-l_2)a} + \frac{r^{(2p+q-l_2)a}}{1-r^{qa}} \\ & \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ & + 1 + r^{qa} + \ldots + r^{\{(q-1)p-l_{q-1}\}a} + \frac{r^{\{(q-1)p+q-l_{q-1}\}a}}{1-r^{qa}}, \end{aligned}$$

and similarly for $(q-1)\,\Sigma\,\dfrac{1}{1-r^{q\beta}}$. Therefore

$$(q-1)\,H = \Sigma_0^{\frac{p-l_1}{q}}\, a/p + \Sigma_0^{\frac{2p-l_2}{q}}\, a/p + \dots + \Sigma_0^{\frac{(q-1)p-l_{q-1}}{q}}\, a/p$$

$$= \Sigma_0^{\frac{p}{q}} a/p + \Sigma_0^{\frac{2p}{q}} a/p + \dots + \Sigma_0^{\frac{(q-1)p}{q}} a/p$$

$$= (q-1)\,\Sigma_0^{\frac{p}{q}} a/p + (q-2)_{\frac{p}{q}}^{\frac{2p}{q}} a/p + \dots + 1\Sigma_{\frac{(q-2)p}{q}}^{\frac{(q-1)p}{q}} a/p$$

$$= (q-1)\,\Sigma_0^{\frac{p}{q}} a/p + (q-3)\,\Sigma_{\frac{p}{q}}^{\frac{2p}{q}} a/p + (q-5)\,\Sigma_{\frac{2p}{q}}^{\frac{3p}{q}} a/p + \dots,$$

the series terminating with the last positive coefficient. In the same way if $q/p = -1$, we get

$$\Sigma\,\frac{1 + r^{-a} + r^{-2a} + \dots + r^{-qa}}{1 - r^{-qa}} = 0,$$

and $(q+1)\,H =$ similar expressions to the above. Therefore generally

$$(q - q/p)\,H = (q-1)\,\Sigma_0^{\frac{p}{q}} a/p + (q-2)\,\Sigma_{\frac{p}{q}}^{\frac{2p}{q}} a/p + \dots + 1\Sigma_{\frac{(q-2)p}{q}}^{\frac{(q-1)p}{q}} a/p$$

$$= (q-1)\,\Sigma_0^{\frac{p}{q}} a/p + (q-3)\,\Sigma_{\frac{p}{q}}^{\frac{2p}{q}} a/p + \dots,$$

where $\Sigma_{\frac{(r-1)p}{q}}^{\frac{rp}{q}} a/p =$ sum of the quadratic characters of the integers between $\dfrac{(r-1)p}{q}$ and $\dfrac{rp}{q}$.

It is usual to express h in terms of the quadratic characters of integers between 0 and $\frac{1}{2}p$; the above formulæ enable us to subdivide and, to some extent, narrow these limits.

2. If $q=3$, the expression on the right-hand side of the last equation reduces to a single term, and we get

$$(3-3/p)\,H = 2\Sigma_0^{\frac{1}{2}p}\, a/p.$$

The value of H may also be reduced to a single term for $q=4$.

For, from the general formula,

$$3H = 3\Sigma_0^{\frac{1}{4}p}\, a/p + 1\Sigma_{\frac{1}{4}p}^{\frac{1}{2}p}\, a/p.$$

But if $p = 8n+7$,

$$3H = 3\Sigma_0^{\frac{1}{4}p}\, a/p + 3\Sigma_{\frac{1}{4}p}^{\frac{1}{2}p}\, a/p,$$

therefore
$$h = H = \Sigma_0^{\frac{1}{4}p}\, a/p,$$

and
$$\Sigma_{\frac{1}{4}p}^{\frac{1}{2}p}\, a/p = 0.$$

For $p = 8n+3$,

$$3H = \Sigma_0^{\frac{1}{4}p}\, a/p + \Sigma_{\frac{1}{4}p}^{\frac{1}{2}p}\, a/p,$$

therefore
$$h = 3H = \Sigma_{\frac{1}{4}p}^{\frac{1}{2}p}\, a/p,$$

and
$$\Sigma_0^{\frac{1}{4}p}\, a/p = 0.$$

3. Taking in succession $q = 2, 3, 6$, we get three equations between H and the sum of the quadratic characters for the intervals 0 to $\frac{1}{6}p$, $\frac{1}{6}p$ to $\frac{1}{3}p$, $\frac{1}{3}p$ to $\frac{1}{2}p$. They are

$$(2-2/p)\,H = \Sigma_0^{\frac{1}{2}p}\, a/p = \Sigma_0^{\frac{1}{6}p}\, a/p + \Sigma_{\frac{1}{6}p}^{\frac{1}{3}p}\, a/p + \Sigma_{\frac{1}{3}p}^{\frac{1}{2}p}\, a/p$$

$$(3-3/p)\,H = 2\Sigma_0^{\frac{1}{3}p}\, a/p = 2\Sigma_0^{\frac{1}{6}p}\, a/p + 2\Sigma_{\frac{1}{6}p}^{\frac{1}{3}p}\, a/p$$

$$(6-6/p)\,H = \qquad = 5\Sigma_0^{\frac{1}{6}p}\, a/p + 3\Sigma_{\frac{1}{6}p}^{\frac{1}{3}p}\, a/p + 1\Sigma_{\frac{1}{3}p}^{\frac{1}{2}p}\, a/p,$$

from which we get

$$H = \frac{2\Sigma_0^{\frac{1}{6}p}\, a/p}{1+2/p+3/p-6/p},$$

Each interval may be expressed in terms of H. For convenience write a_r for $\Sigma_{\frac{1}{6}(r-1)p}^{\frac{1}{6}rp}\, a/p$ and e_n for n/p. The results are given in the following table.

$e_2=$	$+1$	-1	-1	$+1$
$e_3=$	-1	$+1$	-1	$+1$
$a_1=$	H	H	$-H$	H
$a_2=$	H	0	$3H$	0
$a_3=$	$-H$	$2H$	H	0

4. Taking in succession $q=2, 4, 8$, we get three equations between H and the first four 8^{th}-intervals $\Sigma_0^{\frac{1}{8}p} a/p$, $\Sigma_{\frac{1}{8}p}^{\frac{1}{4}p} a/p$, &c.

Writing a_r for the r^{th} interval, we get

for $e_2=+1$, $\qquad a_1+a_2=H,\ \ a_2=a_3,\ \ a_3+a_4=0,$

" $e_2=-1$, $\qquad a_1+a_2=0,\ \ a_1=a_4,\ \ a_2+a_4=3H.$

5. Taking in succession $q=2, 3, 4, 6, 12$, we get five equations between H and the first six 12^{th} intervals. The results obtained are given in the following table.

$e_2=$	$+1$	-1	-1	$+1$
$e_3=$	-1	$+1$	-1	$+1$
$a_1+a_2=$	H	H	$-H$	H
$a_3=$	0	$-H$	H	0
$a_4=$	H	H	$2H$	0
$a_5+a_6=$	$-H$	$2H$	H	0
$a_1+a_5=$	0	H	$2H$	H

6. From this table, we may deduce that

$$h \text{ is not greater than } \frac{p+5}{12} \text{ if } p = 24n+7,$$

$$\text{,,} \quad \text{,,} \quad \text{,,} \quad \frac{p+1}{4} \text{ ,, } p = 24n+11,$$

$$\text{,,} \quad \text{,,} \quad \text{,,} \quad \frac{p+5}{8} \text{ ,, } p = 24n+19,$$

$$\text{,,} \quad \text{,,} \quad \text{,,} \quad \frac{p-5}{6} \text{ ,, } p = 24n+23.$$

7. If in the equation

$$(q - q/p)\, H = (q-1)\, \Sigma_0^{\frac{p}{q}} a/p + (q-3)\, \Sigma_{\frac{p}{q}}^{\frac{2p}{q}} a/p + \ldots,$$

we put $q = p - 1$, a non-residue, then each interval contains one integer only, and we get

$$pH = (p-2)\, 1/p + (p-4)\, 2/p + \ldots + 1 \,.\, \frac{p-1}{2}/p.$$

But, if $p = 8n + 7$, we have also

$$pH = p \,.\, 1/p + p \,.\, 2/p + \ldots + p \,.\, \frac{p-1}{2}/p,$$

therefore

$$0 = 2\left\{1 \,.\, 1/p + 2 \,.\, 2/p + \ldots + \frac{p-1}{2} \,.\, \frac{p-1}{2}/p\right\},$$

or $\Sigma\alpha = \Sigma\beta$, where α and β are the integers less than $\frac{1}{2}p$, such that $\alpha/p = +1$, $\beta/p = -1$, and $\gamma/p = 0$, if γ is not prime to p.

If $p = 8n + 3$, we have also

$$3pH = p \,.\, 1/p + p \,.\, 2/p + \ldots + p \,.\, \frac{p-1}{2}/p,$$

therefore

$$2pH = 2\left\{1 \,.\, 1/p + 2 \,.\, 2/p + \ldots + \frac{p-1}{2} \,.\, \frac{p-1}{2}/p\right\},$$

therefore $pH = \Sigma\alpha - \Sigma\beta$, where α and β are as above defined.

Putting $q = p - 1$ in the equation

$$(q - q/p)\, H = (q-1)\, \Sigma_0^{\frac{p}{q}} a/p + (q-2)\, \Sigma_{\frac{p}{q}}^{\frac{2p}{q}} a/p + \ldots + 1\, \Sigma_{\frac{(q-2)p}{q}}^{\frac{(q-1)p}{q}} a/p,$$

we should get the well-known relation

$$pH = \Sigma\beta - \Sigma\alpha,$$

where the integers range from 1 to $(p-1)$.

8. Putting in succession $q = 2,\ n,\ 2n$, we get

$$(2 - e_2)\, H = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n,$$

$$(n - e_n)\, H = (n-1)\, a_1 + (n-1)\, a_2 + (n-3)\, a_3 + \ldots + a_{n-1} + a_n,$$

$$(2n - e_2 e_n)\, H = (2n-1)\, a_1 + (2n-3)\, a_3 + (2n-5)\, a_5 + \ldots + 3a_{n-1} + a_n,$$

where
$$a_r = \Sigma_{\frac{(r-1)p}{2n}}^{\frac{rp}{2n}} a/p.$$

It is easily deduced that if $e_n = +1$,

$$\left.\begin{aligned} a_1 + a_3 + a_5 + \ldots &= h \\ a_2 + a_4 + a_6 + \ldots &= 0 \end{aligned}\right\},$$

whilst if $e_n = -1$,

$$\left.\begin{aligned} a_1 + a_3 + a_5 + \ldots &= 0 \\ a_2 + a_4 + a_6 + \ldots &= h \end{aligned}\right\}.$$

In each case the series does not proceed beyond the first n of the $2n^{\text{th}}$-intervals.

Thus, if $q = 8$, since $4/p$ is always $+1$,

$$a_1 + a_3 = h,$$

and
$$a_2 + a_4 = 0.$$

9. To prove that, when p is a prime of form $4n + 3$,

$$H = \frac{(p-1)(p-2)}{6} - 2\Sigma_1^{\frac{1}{4}(p-3)} [\sqrt{(np)}],$$

where $[\sqrt{(np)}]$ denotes the integral part of $\sqrt{(np)}$; n assuming all integral values included between the limits 1 and $\frac{1}{4}(p-3)$.

Writing

$$1^2 = \alpha_1 + p\left[\frac{1^2}{p}\right]$$

$$2^2 = \alpha_2 + p\left[\frac{2^2}{p}\right]$$

$$\vdots \qquad\qquad \vdots$$

$$\left(\frac{p-1}{2}\right)^2 = \alpha_{\frac{1}{2}(p-1)} + p\left(\frac{\left(\frac{p-1}{2}\right)^2}{p}\right);$$

then, changing any residue α, which is greater than $\frac{1}{2}(p-1)$ into $p-\beta$, we should have, supposing there are b such residues,

$$\Sigma_1^{\frac{1}{2}(p-1)} n^2 = (\Sigma\alpha - \Sigma\beta) + pb + p\Sigma_1^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right],$$

where α and β are the quadratic residues and non-residues of p less than $\frac{1}{2}p$.

For $p = 8n+7$, $\qquad \Sigma\alpha - \Sigma\beta = 0$,

and therefore, dividing by p,

$$\frac{p^2-1}{24} = b + \Sigma_1^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right],$$

$$\frac{p^2-1}{24} = \frac{p-1-2h}{4} + \Sigma_1^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right],$$

which reduces to

$$h = H = 2\Sigma_1^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right] - \frac{(p-1)(p-5)}{12}.$$

For $p = 8n+3$, $\qquad \Sigma\alpha - \Sigma\beta = pH$,

so that $$\frac{p^2-1}{24} = H + \frac{p-1-2h}{4} + \Sigma_1^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right]$$

$$= -\frac{H}{2} + \frac{p-1}{4} + \Sigma_1^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right],$$

or $$H = 2\Sigma_1^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right] - \frac{(p-1)(p-5)}{12}.$$

The series $\Sigma_1^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right]$ may be changed into a more workable form, by treating it in the manner used for the series $\Sigma\left[\frac{np}{q}\right]$ in the third proof, by Gauss, of the Law of Reciprocity (Mathews, *Theory of Numbers*, Part I., p. 41).

For no term $\left[\frac{(n+1)^2}{p}\right]$ can exceed the preceding term by more than unity, since $(n+1)$ is not greater than $\frac{1}{2}(p-1)$, and if

$$\left[\frac{(n+1)^2}{p}\right]=\left[\frac{n^2}{p}\right]+1,$$

there will be a positive integer t such that

$$\frac{n^2}{p}<t<\frac{(n+1)^2}{p},$$

or
$$n<\sqrt{(pt)}<n+1,$$

and therefore
$$n=[\sqrt{(pt)}],$$

so that $\left[\frac{n^2}{p}\right]=t-1$ for $[\sqrt{(pt)}]-[\sqrt{\{p(t-1)\}}]$ values. Therefore

$$\begin{aligned}\Sigma_1^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right]&=1\{[\sqrt{(2p)}]-[\sqrt{p}]\}\\&+2\{[\sqrt{(3p)}]-[\sqrt{(2p)}]\}\\&\quad\vdots\qquad\qquad\vdots\\&+\frac{p-3}{4}\left\{\left[\sqrt{\left(\frac{p+1}{4}\right)p}\right]-\left[\sqrt{\left(\frac{p-3}{4}\right)p}\right]\right\},\end{aligned}$$

since
$$\frac{p(p-3)}{4}<\left(\frac{p-1}{2}\right)^2<\frac{p(p+1)}{4}.$$

Therefore, arranging terms,

$$\begin{aligned}\Sigma_1^{\frac{1}{2}(p-1)}\left[\frac{n^2}{p}\right]&=\frac{p-3}{4}\left[\sqrt{\left(\frac{p+1}{4}\right)p}\right]\\&\quad-\left\{[\sqrt{p}]+[\sqrt{(2p)}]+\ldots+\left[\sqrt{\left(\frac{p-3}{4}.p\right)}\right]\right\}\\&=\frac{p-3}{4}.\frac{p-1}{2}-\Sigma_1^{\frac{1}{4}(p-3)}[\sqrt{(np)}].\end{aligned}$$

Therefore

$$H = 2 \cdot \frac{p-3}{4} \cdot \frac{p-1}{2} - \frac{(p-1)(p-5)}{12} - 2\Sigma_1^{\frac{1}{4}(p-3)}[\sqrt{(np)}],$$

$$H = \frac{(p-1)(p-2)}{6} - 2\Sigma_1^{\frac{1}{4}(p-3)}[\sqrt{(np)}].*$$

Shrewsbury School.

ON VARIOUS EXPRESSIONS FOR h, THE NUMBER OF PROPERLY PRIMITIVE CLASSES FOR ANY NEGATIVE DETERMINANT, NOT INVOLVING A SQUARE FACTOR.

(Third Paper.)

By *H. Holden.*

1. The following results are obtained:

(1) For $-D = p = 4n+1$,

$$h = \Sigma(-1)^{\frac{1}{2}(n-1)} n/p,$$

$$h = \Sigma(-1)^{\frac{1}{8}(n^2-1)} n/p.$$

(2) For $-D = p = 4n+3$,

$$h = \Sigma(-1)^{\frac{1}{2}(n-1)} n/p,$$

$$h = \Sigma(-1)^{\frac{1}{2}(n-1)+\frac{1}{8}(n^2-1)} n/p.$$

(3) For $-D = 2p = 2(4n+1)$ or $2(4n+3)$,

$$h = \Sigma(-1)^{\frac{1}{8}(n^2-1)} n/p,$$

$$h = \Sigma(-1)^{\frac{1}{2}(n-1)+\frac{1}{8}(n^2-1)} n/p,$$

where, in each case, the summations extend over all positive odd integers less than $-D$.

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* The relation

$$h = \frac{p-1}{2} - 2\Sigma_1^{\frac{1}{4}(p-3)}(-1)^n\left[\sqrt{\left(\frac{np}{2}\right)}\right]$$

may be readily deduced from the fact that, if any residue be less than $\frac{1}{2}p$, the square congruent to it lies between np and $(n+\frac{1}{2})p$.

(4) For $-D=p=4n+1$ or $4n+3$,

$$h=1/p-2/p-3/p+4/p+\ldots \qquad \text{up to } \frac{p-1}{2}/p,$$

$$ph=1.1/p-2.2/p-3.3/p+4.4/p+\ldots \text{ up to } (p-1)/p.$$

(5) For any negative determinant not involving a square factor, $h=\Sigma 2p/n$, where n is any odd positive integer less than $-D$.

2. For $-D=p=4n+3$ it is shown in the preceding paper (p. 103) that

$$(q-q/p)H=(q-1)\,\Sigma_0^{\frac{p}{q}}n/p+(q-2)\,\Sigma_{\frac{p}{q}}^{\frac{2p}{q}}n/p+\ldots+1\Sigma_{\frac{(q-2)p}{q}}^{\frac{(q-1)p}{q}}n/p,$$

where $$H=\frac{h}{2-2/p}.$$

Multiplying both sides by q/p,

$$(q.q/p-1)H=(q-1)\,\Sigma qn/p+(q-2)\,\Sigma(qn+r)/p$$
$$+(q-3)\,\Sigma(qn+2r)/p+\ldots,$$

where $p+r\equiv 0$ (mod. q): the multiples of r being reduced to their least positive residues (mod. q), and the summations include all positive integers less than p.

If $rs\equiv 1$ (mod. q) or $sp+1\equiv 0$ (mod. q), we get

$$(q.q/p-1)H=(q-1)\,\Sigma qn/p$$
$$+(q-1-s)\,\Sigma(qn+1)\,p+(q-1-2s)\,\Sigma(qn+2)/p+\ldots$$

the multiples of s being reduced to their least positive residues (mod. q).

This equation may be used to get relations between the values of $\Sigma(qn+x)/p$.

Thus, if $q=4$, $s=1$, and $4/p=1$, so that

$$3H=3\Sigma 4n/p+2\Sigma(4n+1)/p+1\Sigma(4n+2)/p$$
$$=3\Sigma 4n/p+1.\Sigma(4n+1)/p.$$

Similarly, if $q=2$,

$$(2.2/p-1)H=1\Sigma 2n/p$$
$$=\Sigma 4n/p+\Sigma(4n+2)/p$$
$$=\Sigma 4n/p-\Sigma(4n+1)/p.$$

Combining these two equations, we get, if $2/p = 1$ or $p = 8n + 7$,

$$\Sigma 4n/p \quad = h = -\Sigma(4n+3)/p,$$
$$\Sigma(4n+1)/p = 0 = \quad \Sigma(4n+2)/p,$$

and, if $2/p = -1$ or $p = 8n + 3$,

$$\Sigma 4n/p \quad = 0 = \quad \Sigma(4n+3)/p,$$
$$\Sigma(4n+1)/p = h = -\Sigma(4n+2)/p.$$

3. The results for $q = 6$ are given in the following table:

	$p = 24n + 7$	$p = 24n + 11$	$p = 24n + 19$	$p = 24n + 23$
$\Sigma 6n/p$	$-H$	$-H$	$-H$	H
$\Sigma(6n+1)/p$	H	0	H	0
$\Sigma(6n+2)/p$	H	$-2H$	$-3H$	0
$\Sigma(6n+3)/p$	$-H$	$2H$	$-H$	0
$\Sigma(6n+4)/p$	H	0	H	0
$\Sigma(6n+5)/p$	$-H$	H	$3H$	$-H$

In each case $\Sigma(6n+1)/p = \Sigma(6n+4)/p$.

4. Similarly, if $p = 8n + 7$,

$$h = \Sigma 8n/p + \Sigma(8n+1)/p,$$
$$\Sigma(8n+1)/p = \Sigma(8n+2)/p,$$
$$\Sigma(8n+2)/p + \Sigma(8n+3)/p = 0;$$

and, if $p = 8n + 3$,

$$\Sigma 8n/p + \Sigma(8n+5)/p = 0,$$
$$\Sigma 8n/p = \Sigma(8n+7)/p,$$
$$\Sigma(8n+2)/p + \Sigma(8n+7)/p = -h;$$

therefore generally for $p = 4n + 3$,

$$h = \Sigma(8n+1)/p - \Sigma(8n+7)/p,$$
$$0 = \Sigma(8n+3)/p - \Sigma(8n+5)/p,$$

or $$h = \Sigma(-1)^{\frac{1}{2}(n-1)} n/p = \Sigma(-1)^{\frac{1}{2}(n-1)+\frac{1}{8}(n^2-1)} n/p.$$

The relation $h = \Sigma(-1)^{\frac{1}{2}(n-1)} n/p$ may be changed to

$$h = 1/p - 2/p - 3/p + 4/p + \ldots \pm \frac{p-1}{2}/p,$$

where n/p has a positive sign if $n \equiv 0$ or $1 \pmod{4}$, and has a negative sign if $n \equiv 2$ or $3 \pmod{4}$. Then, since in this series a/p and $(p-a)/p$ are prefixed by different signs, for if $a \equiv 0$ or $1 \pmod{4}$, $p - a \equiv 3$ or $2 \pmod{4}$, and as $a/p = -(p-a)p$ we have

$$ph = 1.1/p - 2.2/p - 3.3/p + 4.4/p + \ldots - (p-1).(p-1)/p.$$

5. For $-D = p = 4n + 1$, a result obtained by Schemmel* may be reduced to

$$\Sigma \frac{1}{1 - ir^\alpha} - \Sigma \frac{1}{1 - ir^\beta} = i\sqrt{p}.h.$$

Now $$\frac{1}{1 - ir^\alpha} = \frac{1}{1-i}\left\{\frac{1 - (ir^\alpha)^p}{1 - ir^\alpha}\right\}$$

$$= \frac{1}{1-i}\{1 + ir^\alpha - r^{2\alpha} - ir^{3\alpha} + r^{4\alpha} + \ldots + r^{(p-1)\alpha}\},$$

and similarly for $\frac{1}{1 - ir^\beta}$;

therefore $$\Sigma \frac{1}{1 - ir^\alpha} - \Sigma \frac{1}{1 - ir^\beta}$$

$$= i\sqrt{p}.h = \frac{\sqrt{p}}{1-i}\{i1/p - 2/p - i3/p + 4/p + \ldots + (p-1)/p\}$$

$$= \frac{1+i}{1-i}.\sqrt{p}\left\{1/p - 2/p - 3/p + 4/p + \ldots \text{ up to } \frac{p-1}{2}/p\right\},$$

or $$h = 1/p - 2/p - 3/p + 4/p + \ldots \text{ up to } \frac{p-1}{2}/p.$$

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* Quoted in Bachmann's *Zahlentheorie*, Theil 2. He gets results which are equivalent to

$$\Sigma \frac{1}{1 - ir^\alpha} - \Sigma \frac{1}{1 - ir^\beta} = i\sqrt{p}.h \text{ for } p = 4n+1 \text{ or } 4n+3,$$

$$\Sigma \frac{1}{1 + ir^\alpha} - \Sigma \frac{1}{1 + ir^\beta} = -i\sqrt{p}\,h \text{ for } p = 4n+1.$$

$$= +i\sqrt{p}.h \text{ for } p = 4n+3.$$

It is not difficult to deduce these values of h from those given by Mathews, *Theory of Numbers*, Part I., pp. 240–242.

This is equivalent to $h=\Sigma(-1)^{\frac{1}{2}(n-1)}n/p$, and, as before, it is easily shown that

$$ph=1.1/p-2.2/p-3.3/p+4.4/p+\ldots+(p-1).(p-1)/p.$$

6. The value of h for $-D=p=4n+1$ may be got in a different form.

Let θ be a primitive root of $x^8=1$. Then, if $2/p=1$, $p=8n+1$,

$$\Sigma\frac{1}{1-ir^a}=\Sigma\frac{1}{1-ir^{2a}}=\tfrac{1}{2}\Sigma\left\{\frac{1}{1-\theta r^a}+\frac{1}{1+\theta r^a}\right\}.$$

Now $$\Sigma\frac{\theta(1-\theta)}{1-\theta r^a}=\Sigma\frac{\theta\{1-(\theta r^a)^p\}}{1-\theta r^a}$$

$$=\Sigma\{\theta+\theta^2r^a+\theta^3r^{2a}-r^{3a}-\theta r^{4a}-\theta^2r^{5a}-\theta^3r^{6a}+r^{7a}+\ldots+\theta r^{(p-1)a}\}$$

$$=\Sigma\{\theta+(\theta^2+\theta)(r^a-r^{4a}-r^{5a}+r^{8a}+\ldots)$$
$$+(\theta^3+1)(r^{2a}-r^{3a}-r^{6a}+r^{7a}+\ldots)\},$$

where the powers of r are less than $\dfrac{pa}{2}$.

Therefore

$$\Sigma\frac{1}{1-\theta r^a}=\Sigma\left\{\frac{\theta}{\theta-\theta^2}+\frac{\theta+\theta^2}{\theta-\theta^2}(r^a-r^{4a}-r^{5a}+r^{8a}+\ldots)\right.$$
$$\left.+\frac{\theta^3+1}{\theta-\theta^2}(r^{2a}-r^{3a}-r^{6a}+r^{7a}+\ldots)\right\}.$$

Therefore

$$\Sigma\frac{1}{1-\theta r^a}-\Sigma\frac{1}{1-\theta r^\beta}=\sqrt{p}\left\{\frac{\theta+\theta^2}{\theta-\theta^2}(1/p-4/p-5/p+8/p+\ldots)\right.$$
$$\left.+\frac{\theta^3+1}{\theta-\theta^2}(2/p-3/p-6/p+7/p+\ldots)\right\}.$$

Similarly,

$$\Sigma\frac{1}{1+\theta r^a}-\Sigma\frac{1}{1+\theta r^\beta}=\sqrt{p}\left\{\frac{\theta-\theta^2}{\theta+\theta^2}(1/p-4/p-5/p+8/p+\ldots)\right.$$
$$\left.+\frac{\theta^3-1}{\theta+\theta^2}(2/p-3/p-6/p+7/p+\ldots)\right\}.$$

Adding and noting that

$$\frac{\theta+\theta^2}{\theta-\theta^2}+\frac{\theta-\theta^2}{\theta+\theta^2}=\frac{\theta^3+1}{\theta-\theta^2}+\frac{\theta^3-1}{\theta+\theta^2}=2i,$$

we have

$$\Sigma \frac{1}{1-ir^{\alpha}} - \Sigma \frac{1}{1-ir^{\beta}} = i\sqrt{p}h$$

$$= \tfrac{1}{2}.2i\sqrt{p}\,(1/p + 2/p - 3/p - 4/p - 5/p - 6/p + 7/p + 8/p + \ldots),$$

or $\quad h = 1/p + 2/p - 3/p - 4/p - 5/p - 6/p + 7/p + 8/p + \ldots$

up to $$\frac{p-1}{2}/p.$$

This is equivalent to

$$h = \Sigma\,(8n+1)/p - \Sigma\,(8n+3)/p - \Sigma\,(8n+5)/p + \Sigma\,(8n+7)/p$$

numbers up to p being taken,

$$= \Sigma\,(-1)^{\frac{1}{8}(n^2-1)}\,n/p.$$

Again, if $2/p = -1$, $p = 8n+5$,

$$\Sigma \frac{1}{1-ir^{\alpha}} = \Sigma \frac{1}{1-ir^{2\beta}} = \tfrac{1}{2}\Sigma \left\{\frac{1}{1-\theta r^{\beta}} + \frac{1}{1+\theta r^{\beta}}\right\},$$

therefore

$$\Sigma \frac{1}{1-ir^{\alpha}} - \Sigma \frac{1}{1-ir^{\beta}} = -\tfrac{1}{2}\Sigma\left\{\frac{1}{1-\theta r} - \frac{1}{1-\theta r^{\beta}} + \frac{1}{1+\theta r^{\alpha}} - \frac{1}{1+\theta r^{\beta}}\right\}.$$

Then, as $1-(\theta r^{\alpha})^{p} = 1 - \theta^5 = 1+\theta$, we get

$$\Sigma \frac{1}{1-\theta r^{\alpha}} - \Sigma \frac{1}{1-\theta r^{\beta}}$$

$$= \sqrt{p}\left\{\frac{\theta-1}{\theta+1}(1/p + 4/p - 5/p - 8/p + \ldots) + \frac{\theta^2+\theta^3}{\theta+1}(2/p + 3/p - 6/p - 7/p + \ldots)\right\}.$$

And

$$\Sigma \frac{1}{1+\theta r^{\alpha}} - \Sigma \frac{1}{1+\theta r^{\beta}}$$

$$= \sqrt{p}\left\{\frac{\theta+1}{\theta-1}(1/p + 4/p - 5/p - 8/p + \ldots) + \frac{\theta^2-\theta^3}{\theta-1}(2/p + 3/p - 6/p - 7/p + \ldots)\right\}.$$

Therefore, since

$$\frac{\theta-1}{\theta+1} + \frac{\theta+1}{\theta-1} = -2i,$$

and $$\frac{\theta^2+\theta^3}{\theta+1} + \frac{\theta^2-\theta^3}{\theta-1} = 2i,$$

we get

$$\iota\sqrt{p}\,.\,h = -\tfrac{1}{2}\,(-2i\sqrt{p})(1/p - 2/p - 3/p + 4/p - 5/p + 6/p + 7p - 8/p + \ldots),$$

therefore

$$h = 1/p - 2/p - 3/p + 4/p - 5/p + 6/p + 7/p - 8/p$$

up to $\dfrac{p-1}{2}/p$.

This is equivalent to

$$h = \Sigma\,(8n+1)/p - \Sigma\,(8n+3)/p - \Sigma\,(8n+5)/p + \Sigma\,(8n+7)/p,$$

the summation including all odd integers less than p.

Therefore $h = \Sigma\,(-1)^{\frac{1}{8}(n^2-1)}\,n/p$ is generally true for $p = 4n+1$.

7. Combining the two results for $-D = p = 4n+1$, we get

$$h = \Sigma\,(8n+1)/p - \Sigma\,(8n+3)/p,$$

$$0 = \Sigma\,(8n+5)/p - \Sigma\,(8n+7)/p,$$

which, expressed as 8[th] intervals, and denoting $\Sigma_{\frac{1}{8}(r-1)p}^{\frac{1}{8}(rp)}\,\mu/p$ by a_r, give for

$$p = 8n+1,\quad h = a_1 - a_3\,;\quad a_2 - a_4 = 0,$$

for

$$p = 8n+5,\quad h = a_2 - a_4\,;\quad a_1 - a_3 = 0.$$

8. For $\quad -D = 2p = 2\,(4n+1)$ or $2\,(4n+3)$

it is known that

$$h = 2\,\{\Sigma\,(8n+1)/p - \Sigma\,(8n+5)/p\},$$

the summations extending over numbers less than p.

It is easily shown that this is equivalent to

$$h = \Sigma\,(8n+1)/p - \Sigma\,(8n+5)/p,$$

where the summations extend over numbers less than $2p$, and that for the same range

$$\Sigma\,(8n+3)/p - \Sigma\,(8n+7)/p = 0.$$

Therefore, for $-D = 2p$,

$$h = \Sigma\,(-1)^{\frac{1}{8}(n^2-1)}\,n/p,$$

$$h = \Sigma\,(-1)^{\frac{1}{2}(n-1)+\frac{1}{8}(n^2-1)}\,n/p,$$

the summation extending over all positive odd integers less than $-D$.

9. Thus for any negative determinant $-D=p$ or $2p$ not involving a square factor

$$h=\Sigma 2p/n,$$

where n is any odd positive integer less than $-D$. In addition

$$h=\Sigma -D/n \text{ for } -D=p=4n+3$$

$$=\Sigma D/n \text{ for the other negative determinants.}$$

For $-D=p=4n+3$ it may be shown from the results in section 4 that

$$\Sigma D/n=-2/p.h.$$

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GROUPS OF SUBTRACTION AND DIVISION.

By *Harold Hilton.*

The following geometrical proof of Dr. Miller's results (*Quarterly Journal*, Vol. XXXVII., pp. 80–87) may be of interest. Draw the circle j whose equation is $x^2+y^2=\chi_2$, and the line l whose equation is $2x=\chi_1$; χ_1 and χ_2 being real. Let P be any real point (x, y), Q its inverse with respect to j, R the inverse (reflexion) of Q in the axis of x, P' the inverse of Q in l; then P' is derived from P by a geometrical operation t consisting of successive inversions in j and l. Now the substitution (s_1) $n'=\chi_1-n$, where $n=x+\sqrt{(-1)}\,y$, replaces R by P'; and the substitution (s_2) $n'=\chi_2\div n$ replaces P by R. Hence s_2s_1 replaces P by P'.

If $4\chi_2 \not> \chi_1^2$, *i.e.* if j and l meet in imaginary points, t is not of finite order. This is easily proved by inverting j and l into concentric circles and remembering that the system formed by a circle and two inverse points inverts into a similar system.

If $4\chi_2>\chi_1^2$, t is of finite order when $\theta\div\pi$ is rational, θ being the angle between l and j. This is proved by inverting j and l with respect to one of their real intersections. Now $\cos\theta=\chi_1\div 2\sqrt{(\chi_2)}$, and therefore $\cos 2\theta$ is rational when χ_1 and χ_2 are rational. Hence by a well-known result 2θ is a multiple of $\frac{1}{3}\pi$, $\frac{1}{2}\pi$, $\frac{2}{3}\pi$, or π. Therefore, if s_2s_1 is of finite order, it is of order 2, 3, 4, or 6, when χ_1 and χ_2 are rational.

A group isomorphic with *any* dihedral group of given order k is evidently obtained by taking $\chi_1\div 2\sqrt{(\chi_2)}=\cos\dfrac{2\pi}{k}$.

EQUALITY OF TWO COMPOUND DETERMINANTS OF ORDERS n AND $n-1$.

By *Thomas Muir, LL.D.*

1. In a recent paper regarding elimination* I was incidentally led to a very interesting identity connecting two compound determinants, one of the nth order with elements of the $(n-1)$th order, and the other of the $(n-1)$th order with elements of the nth order. For the present the general character of the identity may be sufficiently understood from a glance at two of the simplest cases. When $n=3$ it is

$$\begin{vmatrix} |a_1b_2| & |a_1m_2|+|l_1b_2| & |l_1m_2| \\ |a_2b_3| & |a_2m_3|+|l_2b_3| & |l_2m_3| \\ |a_3b_1| & |a_3m_1|+|l_3b_1| & |l_3m_1| \end{vmatrix} = -\begin{vmatrix} |a_1b_2l_3| & |l_1m_2a_3| \\ |a_1b_2m_3| & |l_1m_2b_3| \end{vmatrix},$$

where the elements fundamentally involved are taken from the arrays

$$\begin{matrix} a_1, & a_2, & a_3, & \quad l_1, & l_2, & l_3, \\ b_1, & b_2, & b_3, & \quad m_1, & m_2, & m_3. \end{matrix}$$

When $n=4$ it is

$$\begin{vmatrix} |a_1b_2c_3| & |a_1b_2n_3|+|a_1m_2c_3|+|l_1b_2c_3| & |a_1m_2n_3|+|l_1b_2n_3|+|l_1m_2c_3| & |l_1m_2n_3| \\ |a_2b_3c_4| & |a_2b_3n_4|+|a_2m_3c_4|+|l_2b_3c_4| & |a_2m_3n_4|+|l_2b_3n_4|+|l_2m_3c_4| & |l_2m_3n_4| \\ |a_3b_4c_1| & |a_3b_4n_1|+|a_3m_4c_1|+|l_3b_4c_1| & |a_3m_4n_1|+|l_3b_4n_1|+|l_3m_4c_1| & |l_3m_4n_1| \\ |a_4b_1c_2| & |a_4b_1n_2|+|a_4m_1c_2|+|l_4b_1c_2| & |a_4m_1n_2|+|l_4b_1n_2|+|l_4m_1c_2| & |l_4m_1n_2| \end{vmatrix}$$

$$= -\begin{vmatrix} |a_1b_2c_3l_4| & |a_1m_2c_3l_4|+|a_1b_2n_3l_4| & |a_1m_2n_3l_4| \\ |a_1b_2c_3m_4| & |l_1b_2c_3m_4|+|a_1b_2n_3m_4| & |l_1b_2n_3m_4| \\ |a_1b_2c_3n_4| & |l_1b_2c_3n_4|+|a_1m_2c_3n_4| & |l_1m_2c_3n_4| \end{vmatrix},$$

where the fundamental elements are those of the arrays

$$\begin{matrix} a_1 & a_2 & a_3 & a_4, & \quad l_1 & l_2 & l_3 & l_4, \\ b_1 & b_2 & b_3 & b_4, & \quad m_1 & m_2 & m_3 & m_4, \\ c_1 & c_2 & c_3 & c_4, & \quad n_1 & n_2 & n_3 & n_4. \end{matrix}$$

In the former case, if we make one row of the second array identical with the non-correspondent row of the first, say,

$$l_1, l_2, l_3 = b_1, b_2, b_3,$$

* See *Transactions Royal Soc. Edinburgh*, XLV., pp. 1–7.

we have

$$\begin{vmatrix} |a_1b_2| & |a_1m_2| & |b_1m_2| \\ |a_2b_3| & |a_2m_3| & |b_2m_3| \\ |a_3b_1| & |a_3m_1| & |b_3m_1| \end{vmatrix} = |a_1b_2m_3|^2;$$

and in the latter if we make each of two rows of the second array identical with a non-correspondent row of the first, say,

$$l_1,\ l_2,\ l_3,\ l_4 = b_1,\ b_2,\ b_3,\ b_4,$$
$$m_1,\ m_2,\ m_3,\ m_4 = c_1,\ c_2,\ c_3,\ c_4,$$

we have

$$\begin{vmatrix} |a_1b_2c_3| & |a_1b_2n_3| & |a_1c_2n_3| & |b_1c_2n_3| \\ |a_2b_3c_4| & |a_2b_3n_4| & |a_2c_3n_4| & |b_2c_3n_4| \\ |a_3b_4c_1| & |a_3b_4n_1| & |a_3c_4n_1| & |b_3c_4n_1| \\ |a_4b_1c_2| & |a_4b_1n_2| & |a_4c_1n_2| & |b_4c_1n_2| \end{vmatrix} = -\,|a_1b_2c_3n_4|^3.$$

In other words, with such substitutions the identity degenerates into Cauchy's theorem regarding the relation between a determinant and its adjugate, and may therefore be viewed as a generalisation of the latter.

2. In seeking to obtain a concise proof of this identity there came to light a still more interesting one of a similar kind,—similar, that is to say, in general form—and having a further link of connection in being also viewable as a generalisation of Cauchy's theorem, but reaching this special result in two stages and having as intermediate result a theorem equally important with Cauchy's.

When $n=3$ the fundamental elements are contained in three 2-by-3 arrays, say

$$\begin{matrix} a_1 & a_2 & a_3, \\ b_1 & b_2 & b_3, \end{matrix} \qquad \begin{matrix} h_1 & h_2 & h_3, \\ k_1 & k_2 & k_3, \end{matrix} \qquad \begin{matrix} m_1 & m_2 & m_3, \\ n_1 & n_2 & n_3, \end{matrix}$$

and the identity is

$$\begin{vmatrix} |a_1b_2| & |h_1k_2| & |m_1n_2| \\ |a_2b_3| & |h_2k_3| & |m_2n_3| \\ |a_3b_1| & |h_3k_1| & |m_3n_1| \end{vmatrix} = \begin{vmatrix} |a_1b_2m_3| & |a_1b_2n_3| \\ |h_1k_2m_3| & |h_1k_2n_3| \end{vmatrix};$$

when $n=4$ the fundamental elements are contained in four 3-by-4 arrays, say,

$$\begin{matrix} a_1 & a_2 & a_3 & a_4, \\ b_1 & b_2 & b_3 & b_4, \\ c_1 & c_2 & c_3 & c_4, \end{matrix} \quad \begin{matrix} h_1 & h_2 & h_3 & h_4, \\ k_1 & k_2 & k_3 & k_4, \\ l_1 & l_2 & l_3 & l_4, \end{matrix} \quad \begin{matrix} m_1 & m_2 & m_3 & m_4, \\ n_1 & n_2 & n_3 & n_4, \\ r_1 & r_2 & r_3 & r_4, \end{matrix} \quad \begin{matrix} x_1 & x_2 & x_3 & x_4, \\ y_1 & y_2 & y_3 & y_4, \\ z_1 & z_2 & z_3 & z_4, \end{matrix}$$

and the identity is

$$\begin{vmatrix} |a_1b_2c_3| & |h_1k_2l_3| & |m_1n_2r_3| & |x_1y_2z_3| \\ |a_2b_3c_4| & |h_2k_3l_4| & |m_2n_3r_4| & |x_2y_3z_4| \\ |a_3b_4c_1| & |h_3k_4l_1| & |m_3n_4r_1| & |x_3y_4z_1| \\ |a_4b_1c_2| & |h_4k_1l_2| & |m_4n_1r_2| & |x_4y_1z_2| \end{vmatrix}$$

$$= - \begin{vmatrix} |a_1b_2c_3x_4| & |h_1k_2l_3x_4| & |m_1n_2r_3x_4| \\ |a_1b_2c_3y_4| & |h_1k_2l_3y_4| & |m_1n_2r_3y_4| \\ |a_1b_2c_3z_4| & |h_1k_2l_3z_4| & |m_1n_2r_3z_4| \end{vmatrix}.$$

If in the former case we make the first row of the second and third arrays equal to the second row of the first and second arrays respectively, or, what is the same thing, perform the substitution

$$h,\ k,\ m,\ n\ =\ b,\ c,\ c,\ d,$$

the identity becomes

$$\begin{vmatrix} |a_1b_2| & |b_1c_2| & |c_1d_2| \\ |a_2b_3| & |b_2c_3| & |c_2d_3| \\ |a_3b_1| & |b_3c_1| & |c_3d_1| \end{vmatrix} = |a_1b_2c_3| \cdot |b_1c_2d_3|,$$

which, on putting the d's equal to the a's, degenerates into Cauchy's theorem: and by a similar substitution in the other case we obtain the intermediate result

$$\begin{vmatrix} |a_1b_2c_3| & |b_1c_2d_3| & |c_1d_2e_3| & |d_1e_2f_3| \\ |a_2b_3c_4| & |b_2c_3d_4| & |c_2d_3e_4| & |d_2e_3f_4| \\ |a_3b_4c_1| & |b_3c_4d_1| & |c_3d_4e_1| & |d_3e_4f_1| \\ |a_4b_1c_2| & |b_4c_1d_2| & |c_4d_1e_2| & |d_4e_1f_2| \end{vmatrix} = - |a_1b_2c_3d_4| \cdot |b_1c_2d_3e_4| \cdot |c_1d_2e_3f_4|,$$

from which Cauchy's is derived by putting the e's equal to a's and the f's equal to b's.

3. The mode of formation of the two equivalent determinants in the new identity is readily made clear, and the general result may be formulated thus:

If there be n arrays, each of $n-1$ rows and n columns; and two determinants be formed therefrom, namely, one whose r^{th} column has for its elements the n determinants of the $(n-1)^{\text{th}}$ order derivable from the r^{th} array, and the other whose r^{th} column has for elements the $n-1$ determinants of the n^{th} order formable by adding to the r^{th} array one row of the n^{th} array, the two determinants are equal.

As any one of the n arrays may be made the n^{th}, it is clear that there are n possible forms of the second of the compound determinants composing the identity.

4. The following is probably the simplest possible mode of proof. For convenience in writing it is restricted to the case where $n=4$, and where the identity to be established may be shortly written in the form

$$D_{4,3} = -\Delta_{3,4},$$

the first suffix indicating the order of the compound determinant, and the second suffix the order of each of its elements.

Multiplying $D_{4,3}$ by $-|x_1y_2z_4|$ in the form

$$\begin{vmatrix} x_4 & -x_1 & x_2 & -x_3 \\ y_4 & -y_1 & y_2 & -y_3 \\ z_4 & -z_1 & z_2 & -z_3 \\ . & . & . & 1 \end{vmatrix}$$

we obtain

$$-D_{4,3}\cdot|x_1y_2z_4| = \begin{vmatrix} |a_1b_2c_3x_4| & |a_1b_2c_3y_4| & |a_1b_2c_3z_4| & |a_4b_1c_2| \\ |h_1k_2l_3x_4| & |h_1k_2l_3y_4| & |h_1k_2l_3z_4| & |h_4k_1l_2| \\ |m_1n_2r_3x_4| & |m_1n_2r_3y_4| & |m_1n_2r_3z_4| & |m_4n_1r_2| \\ . & . & . & |x_4y_1z_2| \end{vmatrix},$$

which $$= \Delta_{3,4}\cdot|x_4y_1z_2|,$$

whence at once we have as desired

$$D_{4,3} = \Delta_{3,4}.$$

The same mode of proof is equally suitable for the identity of § 1. It suggests that increased elegance might be attained by enunciating the identities slightly differently, namely, by changing rows into columns in the left-hand members, and following a different principle of arrangement, putting, for example, instead of the column

$$|a_1b_2c_3|,\quad |a_2b_3c_4|,\quad |a_3b_4c_1|,\quad |a_4b_1c_2|,$$

the row

$$|a_2b_3c_4|,\quad |a_1b_3c_4|,\quad |a_1b_2c_4|,\quad |a_1b_2c_3|.$$

6. The correlative identities resulting from the application of the Law of Complementaries and the Law of Extensible Minors need only be referred to.

London, *November* 26*th*, 1905.

ON AN EXPANSION OF AN ARBITRARY FUNCTION IN A SERIES OF BESSEL FUNCTIONS.

By *W. Kapteyn.*

In Vol. XXXIII., p. 55, of the *Messenger*, Mr. H. A. Webb gave an expansion of an arbitrary function in a series of Bessel functions which seems not to possess the generality which the author ascribes to it. For observing that

$$(1) \quad \int_0^\infty \frac{1}{z} J_m(z)\, J_n(z)\, dr = \frac{1}{\pi} \frac{\sin\frac{m-n}{2}\pi}{m^2-n^2}, \quad m \neq n,^*$$

$$(2) \quad \int_0^\infty \frac{1}{z} J_n^2(z)\, dr = \frac{1}{2n}.$$

$I_m(z)$ being the Bessel function of the first kind, and assuming the possibility of the expansion

$$(3) \quad \phi(x) = c_1 J_1(x) + c_3 J_3(x) + c_5 J_5(x) + \ldots,$$

we have

$$c_n = 2n \int_0^\infty \frac{\phi(z)}{z} J_n(z)\, dz \quad (n \text{ odd}).$$

Now the author gives as conditions of the expansibility of $\phi(x)$ in the form (3) (I) the integrals c_n must exist, and (II) the series must be absolutely convergent. Considering however the function $\phi(x) = \sin\left(\frac{x}{\sin\phi}\right)$, the integrals c_n exist, for

$$c_n = 2n \int_0^\infty \frac{\sin\left(\frac{z}{\sin\phi}\right)}{z} J_n(z)\, dz = 2 \sin\frac{n\pi}{2}\left(\operatorname{tg}\frac{\phi}{2}\right)^n. \dagger$$

and the series

$$2\left[\operatorname{tg}\frac{\phi}{2} J_1(x) - \operatorname{tg}^3\frac{\phi}{2} J_3(x) + \operatorname{tg}^5\frac{\phi}{2} J_5(x) - \ldots\right]$$

is absolutely convergent for all real values of x. Nevertheless the equation

$$\sin\left(\frac{x}{\sin\phi}\right) = 2\left[\operatorname{tg}\frac{\phi}{2} J_1(x) - \operatorname{tg}^3\frac{\phi}{2} J_3(x) + \operatorname{tg}^5\frac{\phi}{2} J_5(x) - \ldots\right]$$

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* *Acad. of Sciences, Amsterdam*, X., 1901, p. 113.

† Nielsen, *Handbuch der Cylinderfunctionen*, p. 197.

is impossible, for developing both members in ascending powers of x we obtain different results.

In order to obtain the true conditions of expansibility we may consider the sum of the series

$$c_1 J_1 + c_3 J_3 + c_5 J_5 + \ldots,$$

the coefficients having the values

$$c_n = 2n \int_0^\infty \frac{\phi(\alpha)}{\alpha} J_n(\alpha)\, d\alpha.$$

Let S be the required sum, then we find

$$S = 2\int_0^\infty \frac{\phi(\alpha)}{\alpha} \sum_{1,3}^{\infty} n J_n(\alpha) J_n(x)\, d\alpha.$$

Now the series under the integral sign may be transformed by the identity

$$\sum_{1,3}^{n} n J_n(\alpha) J_n(x) = \frac{\alpha}{4} \int_0^x J_0(x-\gamma) \left[\frac{J_1(\alpha-\gamma)}{\alpha-\gamma} + \frac{J_1(\alpha+\gamma)}{\alpha+\gamma} \right] d\gamma,$$

which we have proved elsewhere.*

Hence $$S = \tfrac{1}{2} \int_0^x J_0(x-\gamma)\, M(\gamma)\, d\gamma,$$

where $$M(\gamma) = \int_0^\infty \phi(\alpha) \left[\frac{J_1(\alpha-\gamma)}{\alpha-\gamma} + \frac{J_1(\alpha+\gamma)}{\alpha+\gamma} \right] d\alpha.$$

Supposing $$\phi(\alpha) = -\phi(-\alpha),$$

the integral $M(\gamma)$ may be written

$$M(\gamma) = \int_0^\infty \frac{J_1(t)}{t} [\phi(t+\gamma) + \phi(t-\gamma)]\, dt + 2\int_0^\gamma \phi(\gamma - t) \frac{J_1(t)}{t}\, dt,$$

therefore

$$S = \tfrac{1}{2} \int_0^x J_0(x-\gamma)\, d\gamma \int_0^\infty \frac{J_1(t)}{t} [\phi(t+\gamma) + \phi(t-\gamma)]\, dt$$
$$+ \int_0^x J_0(x-\gamma)\, d\gamma \int_0^\gamma \phi(\gamma-t) \frac{J_1(t)}{t}\, dt.$$

The latter double integral may be reduced to

$$\int_0^x \phi(x-\beta)\, J_1(\beta)\, d\beta,$$

* *Acad. of Sciences, Amsterdam,* XIII., 1905, p. 482.

for by changing the order of the integrations the double integral takes the form

$$\int_0^x \frac{J_1(t)}{t}\,dt \int_t^x J_0(x-t)\,\phi(\gamma-t)\,dt$$

or, writing $$\gamma = x + t - \beta$$

$$\int_0^x \frac{J_1(t)}{t}\,dt \int_t^x J_0(\beta-t)\,\phi(x-\beta)\,d\beta.$$

If now we change again the order of the two integrations, this double integral reduces to

$$\int_0^x \phi(x-\beta)\,d\beta \int_0^\beta J_0(\beta-t)\,\frac{J_1(t)}{t}\,dt,$$

or, as $$\int_0^\beta J_0(\beta-t)\,\frac{J_1(t)}{t}\,dt = J_1(\beta)\,^*$$

to $$\int_0^x \phi(x-\beta)\,J_1(\beta)\,d\beta = \int_0^x \phi(\gamma)\,J_1(x-\gamma)\,d\gamma.$$

Therefore

$$S = \tfrac{1}{2}\int_0^x J_0(x-\gamma)\,d\gamma \int_0^\infty \frac{J_1(t)}{t}\,[\phi(t+\gamma) + \phi(t-\gamma)]\,dt$$
$$+ \int_0^x \phi(\gamma)\,J_1(x-\gamma)\,d\gamma,$$

and the question arises when the second member of this equation reduces to $\phi(x)$. Now it is evident that this will be the case when

$$\int_0^\infty \frac{J_1(t)}{t}\,[\phi(t+\gamma) + \phi(t-\gamma)]\,dt = 2\,\frac{d\phi(\gamma)}{d\gamma},$$

for then

$$\int_0^x J_0(x-\gamma)\,\frac{d\phi}{d\gamma}\,d\gamma = \phi(x) - \phi(0) - \int_0^x \phi(\gamma)\,J_1(x-\gamma)\,d\gamma,$$

and finally $$S = \phi(x),$$

when $\phi(0) = 0$.

From this discussion we conclude that a given function may be developed in a series of the form (3) when the function

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* *Acad. of Sciences, Amsterdam*, 1905, p. 482.

is uneven, vanishes for $x=0$, is continuous for all real values of the variable, and, moreover, satisfies the condition

$$(4) \qquad \int_0^\infty \frac{J_1(t)}{t}\left[\phi(t+\gamma)+\phi(t-\gamma)\right]dt = 2\frac{d\phi(\gamma)}{d\gamma}.$$

It is evident that

$$\sin(x\sin\phi), \quad x\cos(x\sin\phi), \quad x$$

are examples of functions which satisfy the conditions, but if

$$\phi(x) = \sin\left(\frac{x}{\sin\phi}\right),$$

we have

$$\int_0^\infty \frac{J_1(t)}{t}\left[\phi(t+\gamma)+\phi(t-\gamma)\right]dt$$

$$= 2\cos\left(\frac{\gamma}{\sin\phi}\right)\int_0^\infty \frac{J_1(t)}{t}\sin\left(\frac{t}{\sin\phi}\right)dt$$

$$= 2\cos\left(\frac{\gamma}{\sin\phi}\right)\left[\frac{1}{\sin\phi}-\cot\phi\right]$$

$$= 2\frac{d\phi(\gamma)}{d\gamma} - 2\cot\phi\cos\left(\frac{\gamma}{\sin\phi}\right),$$

and

$$c_n = 2n\int_0^\infty \frac{J_n(z)}{z}\sin\left(\frac{z}{\sin\phi}\right)dz = 2\sin\frac{n\pi}{2}\operatorname{tg}^n\frac{\phi}{2},$$

therefore the series

$$2\left[\operatorname{tg}\frac{\phi}{2}J_1(x) - \operatorname{tg}^3\frac{\phi}{2}J_3(x) + \operatorname{tg}^5\frac{\phi}{2}J_5(x)\ldots\right]$$

does not converge to the value $\phi(x) = \sin\left(\frac{x}{\sin\phi}\right)$, but to the value

$$\sin\left(\frac{x}{\sin\phi}\right) - \cot\phi\int_0^x J_0(x-\gamma)\cos\left(\frac{\gamma}{\sin\phi}\right)d\gamma.$$

It would be an interesting problem to determine all the possible forms of $\phi(x)$ which satisfy the condition (4). As to the expression of Neumann's function

$$O_n(z) = \int_0^\infty \frac{J_n(t)}{t(z-t)}dt,$$

we finally remark that the integral in the second member infinite, z having any positive value.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy*, Trinity College, Cambridge.

XVII.

On the integration of series.

§ 1. There is one case of term-by-term integration of series which is of frequent occurrence in analysis, and is not, to the best of my knowledge, treated adequately in the books. It is proved in the books that if

$$\int_a^A f(x)\,dx$$

is absolutely convergent, and $\phi(x)$ can be expanded in a uniformly convergent series

$$\Sigma\phi_n(x),$$

then*

$$\int (f\Sigma\phi_n)\,dx = \Sigma \int f\phi_n\,dx.$$

But although this criterion works well enough when f becomes infinite within the range of integration and ϕ is continuous, it is useless when it is the factor which we wish to expand which becomes infinite. It does not, for example, justify the assertion that

$$\int_0^1 \frac{x^m\,dx}{\sqrt{(1-x^2)}} = \Sigma\,\frac{1.3\ldots 2n-1}{2.4\ldots 2n}\int_0^1 x^{2n+m}\,dx.$$

My object in this note is to give a simple test which enables us in practice to recognise without difficulty in cases of this kind the legitimacy of the proposed operation.

§ 2. Theorem † *If* (i) $f(x)$ *is continuous throughout* (a, A), (ii) $\phi(x)$ *can be expanded in a series of continuous functions*

$$\phi_0(x) + \phi_1(x) + \phi_2(x) + \ldots$$

uniformly convergent throughout $(a, A-\epsilon)$, *however small be the positive quantity* ϵ, *and* (iii) *the integral*

$$\int_a^A \phi(x)\,dx$$

Stolz, *Grundzüge*, I, p. 478; Dini, *Grundlagen*, p. 578.

† I stated this theorem incidentally and without proof in the *Proc. Lond. Math. Soc.*, Vol. II, p. 407.

is convergent, where

$$\bar{\phi}(x) = |\phi_0(x)| + |\phi_1(x)| + \cdots,$$

then
$$\int_a^A \phi f dx = \Sigma \int_a^A \phi_n f dx.$$

In particular the theorem holds if $\phi_n(x) > 0$, and

$$\int_a^A \phi(x)\, dx$$

is convergent.

It is to be observed carefully that the condition that

$$\int_a^A \phi(x)\, dx$$

is *absolutely* convergent is *not* sufficient to replace (iii). In fact it is well known that the term-by-term integration may well be illegitimate even when conditions (i) and (ii) are satisfied and $\phi(x)$ is continuous, as appears from the classical example in which

$$f(x) = 1,$$

$$\phi(x) = 0,$$

$$\phi_n(x) = 2n^2 x e^{-n^2x^2} - 2(n+1)^2 x e^{-(n+1)^2x^2},$$

and
$$a = 0.$$

It is easy to verify that in this case condition (iii) is not satisfied.

In fact, $\phi_n(x) = 0$ gives

$$n^2 e^{-n^2x^2} = (n+1)^2 e^{-(n+1)^2x^2},$$

$$n^2x^2 - 2\log n = (n+1)^2x^2 - 2\log(n+1),$$

$$(2n+1)x^2 = 2\log(1+1/n),$$

$$x = \sqrt{\left(\frac{\log(1+1/n)}{n+\frac{1}{2}}\right)},$$

which is $< 1/n$. Thus, for values of $x > 1/n$, $\phi_n(x)$ is positive.

Also
$$\int_\epsilon^1 |\phi_n(x)|\, dx > \int_{1/n}^1 \phi_n(x)\, dx$$

if $\epsilon < 1/n$. But

$$\int_{1/n}^1 \phi_n(x)\, dx = \left[-e^{-n^2x^2} + e^{-(n+1)^2x^2}\right]_{1/n}^1$$

$$= -e^{-n^2} + e^{-(n+1)^2} + e^{-1} - e^{-(1+1/n)^2}.$$

Now $\quad e^{-1} - e^{-(1+1/n)^2} = e^{-1}\{1 - e^{-2/n-1/n^2}\} > K/n,$

where K is independent of n. Hence, if $\epsilon < 1/N$,

$$\sum_0^N \int_\epsilon^1 |\phi_n(x)|\,dx > K(1 + \tfrac{1}{2} + \ldots + 1/N) > K \log N,$$

so that
$$\int_\epsilon^1 \bar{\phi}(x)\,dx > K \log N,$$

which can be made as large as we please by sufficiently increasing N. Hence

$$\int_0^1 \bar{\phi}(x)\,dx$$

is not convergent.

§ 3. I proceed now to the proof of the theorem stated in § 2. In the first place it is evident that if $\epsilon > 0$,

$$(1) \qquad \int_a^{A-\epsilon} \phi f dx = \Sigma \int_a^{A-\epsilon} \phi_n f dx.$$

The left-hand side tends for $\epsilon = 0$ to the limit

$$\int_a^A \phi f dx,$$

since f is continuous and $\int \phi dx$ absolutely convergent, because

$$|\phi| \leqq \bar{\phi}.$$

We have therefore to prove that

$$(2) \qquad \lim_{\epsilon=0} \sum_0^\infty \int_a^{A-\epsilon} \phi_n f dx$$

exists and is equal to

$$(3) \qquad \sum_0^\infty \int_a^A \phi_n f dx.$$

Now

$$(4) \qquad \sum_{N_1}^{N_2} \int_a^A \phi_n f dx = \sum_{N_1}^{N_2} \left(\int_a^{A-\delta} + \int_{A-\delta}^A\right) \phi_n f dx,$$

$$\left|\sum_{N_1}^{N_2} \int_a^A \phi_n f dx\right| \leqq \left|\int_a^{A-\delta} f\left(\sum_{N_1}^{N_2} \phi_n\right) dx\right| + \left|\int_{A-\delta}^A f\left(\sum_{N_1}^{N_2} \phi_n\right) dx\right|$$

$$\leqq \left|\int_a^{A-\delta} f\left(\sum_{N_1}^{N_2} \phi_n\right) dx\right| + \int_{A-\delta}^A \bar{\phi}\,|f|\,dx.$$

The second term is less than a constant multiple of $\int_{A-\delta}^{A} \bar{\phi} dx$, and may be made $< \sigma$ by taking δ small enough. Having chosen δ we can choose N_1, so that

$$\left| \sum_{N_1}^{N_2} \phi_n \right| < \sigma$$

for all values of x in $(a, A-\delta)$, and all values of $N_2 \geqq N_1$.

Then
$$\left| \sum_{N_1}^{N_2} \int_a^A \phi_n f dx \right| < K\sigma,$$

where K is a constant. It follows that the series (3) is convergent. Hence

$$\sum_0^\infty \int_{A-\epsilon}^{A} \phi_n f dx$$

is convergent. Its modulus is plainly not greater than a certain constant multiple of

$$\int_{A-\epsilon}^{A} \bar{\phi} dx,$$

and its limit for $\epsilon = 0$ is zero. The theorem is accordingly established.

§ 4. If, *e.g.*,

$$\phi(x) = (1-x)^{-\alpha} \quad (0 < \alpha < 1) \quad a = 0, \ A = 1,$$

$$\phi_n(x) = \frac{\alpha(\alpha+1)\dots(\alpha+n-1)}{1.2\dots n} > 0,$$

so that $\phi(x) \equiv \bar{\phi}(x)$, and the conditions are satisfied. Thus,

$$\int_0^1 (1-x)^{-\alpha} f(x)\, dx = \Sigma \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)} \int_0^1 x^n f(x)\, dx.$$

If $\alpha = \beta + i\gamma$, where $0 < \beta < 1$, ϕ and $\bar{\phi}$ are no longer identical. But it is easy to see that however small be the positive quantity k,

$$|\alpha + \nu| < \beta + \nu + k$$

for all values of ν after a certain finite value. Hence (after a finite number of terms),

$$|\phi_n(x)| < K \frac{(\beta+k)(\beta+k+1)\dots(\beta+k+n-1)}{1.2\dots n},$$

and so
$$\bar{\phi}(x) < K(1-x)^{-\beta-k}.$$

We can choose k so that $\beta + k < 1$. Hence

$$\int_0^1 \bar{\phi}(x)\,dx$$

is convergent. The integration term by term is therefore still valid.

§ 5. An almost obvious generalisation is the following: *if* (i) f *is continuous in* $(a, A-\epsilon)$, *and* $\Sigma\phi_n$ *uniformly convergent in* $(a, A-\epsilon)$, *however small be* ϵ, *and* (ii) *the integral*

$$\int_a^A |f|\,\bar{\phi}\,dx$$

is convergent, then the integration term by term is permissible.

Precisely similar reasoning leads to the result that *if* (i) f *is continuous for all finite values of* x, (ii) $\Sigma\phi_n$ *is uniformly convergent in an arbitrary interval* (a, A), *and* (iii)

$$\int_a^\infty |f|\,\bar{\phi}\,dx$$

is convergent, then

$$\int_a^\infty f\Sigma\phi_n\,dx = \Sigma\int_a^\infty f\phi_n\,dx.$$

For example, take the case in which the lower limit is 0, $f = e^{-ax^2}$, and $\phi = \cos 2bx$, so that

$$\phi_n = \frac{(-)^n}{2n!}(2bx)^{2n}.$$

Here $$\bar{\phi} = \sum_0^\infty \frac{(2bx)^{2n}}{2n!} = \cosh 2bx,$$

and the conditions are obviously satisfied. If $f = e^{-ax}$, $\phi = \cos bx$, $\bar{\phi} = \cosh bx$, and the conditions are satisfied only if $a > b$. In this case the integral series is

$$\Sigma\frac{(-)^n}{2n!}b^{2n}\int_0^\infty e^{-ax}x^{2n}\,dx = \Sigma(-)^n\frac{b^{2n}}{a^{2n+1}},$$

which is divergent if $b > a$. This last example would make it appear that the limitations of the theorem are not unduly narrow.

ELEMENTARY STANDARD FORMS OF THE INTEGRAL CALCULUS.

By *T. J. I'a. Bromwich*, Queen's College, Galway.

1. It seems somewhat surprising that most of the English text-books on the Integral Calculus give forms for the standard integrals

$$(\alpha)\int_0^x (a^2-x^2)^{-\frac{1}{2}}dx, \qquad (\beta)\int_0^x (a^2+x^2)^{-\frac{1}{2}}dx, \qquad (\gamma)\int^x (x^2-a^2)^{-\frac{1}{2}}dx$$

which are not wholly accurate.* The forms usually given are

$$(\alpha)\ \sin^{-1}(x/a), \quad (\beta)\ \sinh^{-1}(x/a), \quad (\gamma)\ \cosh^{-1}(x/a),$$

which are only correct if a is positive for (α), (β), and if x/a is positive for (γ).

As a matter of fact the formulæ should be

$$(\alpha)\ \sin^{-1}(x/|a|),$$

$$(\beta)\ \sinh^{-1}(x/|a|) = \log[\{x+(a^2+x^2)^{\frac{1}{2}}\}/|a|],$$

$$(\gamma)\ \pm\cosh^{-1}(|x|/|a|) = \log[|x+(x^2-a^2)^{\frac{1}{2}}|].$$

where in (γ) the sign $\pm$ is to be the same as the sign of x.

Of course, in most applications of these forms the sign of a can be foreseen, and it is then generally taken positively; but it occasionally happens that the sign of a depends on other numbers, and in such cases errors may occur through not emphasising the proper sign. Thus the integral

$$\int_0^a (a^2-x^2)^{-\frac{1}{2}}dx$$

would be taken as $\sin^{-1}(1) = \frac{1}{2}\pi$ by the ordinary formula; but its true value is $\sin^{-1}(\pm 1) = \pm\frac{1}{2}\pi$, where the ambiguous sign is the same as the sign of a.

It is often better to use the standard forms, of which the first is given by Stolz:

$$(\alpha)\ \tan^{-1}(x/y), \quad y=(a^2-x^2)^{\frac{1}{2}},$$

$$(\beta)\ \tanh^{-1}(x/y), \quad y=(a^2+x^2)^{\frac{1}{2}},$$

$$(\gamma)\ \tanh^{-1}(y/x), \quad y=(x^2-a^2)^{\frac{1}{2}}.$$

* We assume that the square-roots are real and *positive*; and that the angles lie between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

From (γ) we can infer the obvious corollary that $\int_0^\theta \sec\theta\, d\theta$, which is frequently given as $\cosh^{-1}(\sec\theta)$, should really be

$$\pm\cosh^{-1}(\sec\theta) = \tanh^{-1}(\sin\theta) = \log(|\sec\theta + \tan\theta|),$$

where θ lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

2. The standard forms referred to in § 1 will cover all cases of the type

$$\int_{x_0}^{x_1} dx/y,\quad y^2 = ax^2 + 2bx + c$$

by applying a linear transformation to x; in many of the ordinary calculations this method is as simple as any. But it is sometimes more convenient to apply the formula

$$\int_{x_0}^{x_1} \frac{dx}{y} = \frac{2}{\alpha}\tanh^{-1}\frac{\alpha(x_1 - x_0)}{y_1 + y_0},\ \text{if } a = \alpha^2 > 0;$$

$$\text{or } \frac{2}{\beta}\tan^{-1}\frac{\beta(x_1 - x_0)}{y_1 + y_0},\ \text{if } a = -\beta^2 < 0.$$

This result does not appear to have been given in any text-book, and it is not very easy to deduce from the previous formulae; the following investigation is simple and straightforward, although several other more or less obvious methods can easily be found.

Let
$$t = \frac{x - x_0}{y + y_0},$$

then
$$y\frac{dt}{dx} = \frac{y}{y + y_0} - \frac{x - x_0}{(y + y_0)^2}\, y\frac{dy}{dx}$$

$$= \frac{y}{y + y_0} - \frac{x - x_0}{(y + y_0)^2}(ax + b)$$

$$= \frac{yy_0 + axx_0 + b(x + x_0) + c}{(y + y_0)^2}.$$

Also,

$$1 - at^2 = 1 - a\frac{(x - x_0)^2}{(y + y_0)^2} = 2\frac{yy_0 + axx_0 + b(x + x_0) + c}{(y + y_0)^2}.$$

Hence $$y\frac{dt}{dx}=\tfrac{1}{2}(1-at^2),$$
so that
$$\int_{x_0}^{x_1}\frac{dx}{y}=2\int_0^{t_1}\frac{dt}{1-at^2}=\frac{2}{\alpha}\tanh^{-1}(\alpha t_1),\ \text{ if } a=\alpha^2,$$
$$\text{or } \frac{2}{\beta}\tan^{-1}(\beta t_1),\ \text{ if } a=-\beta^2.$$

To illustrate the result take the following cases:

(i) $$u=\int_0^1 dx/y,\quad y^2=[1-(1-p^2)x][1-(1-q^2)x].$$

Here $x-x_0=1$, $y+y_0=1+pq$, taking p and q to be positive. Thus if $(1-p^2)(1-q^2)$ is positive,
$$u=\frac{2}{\sqrt{\{(1-p^2)(1-q^2)\}}}\tanh^{-1}\left(\frac{\sqrt{\{(1-p^2)(1-q^2)\}}}{1+pq}\right),$$
$$=2\omega/(p+q)\sinh\omega,\ \text{ where }\ \cosh\omega=(1+pq)/(p+q).$$

When $(1-p^2)(1-q^2)$ is negative, the hyperbolic functions are replaced by circular functions.

This result occurs in calculating the attraction between two halves of a homoeoid.

(ii) $$u=\int_0^{\frac{1}{2}\pi}x(x^2-1)^{-\frac{1}{2}}d\theta,\quad x=\cosh\alpha\cos\theta+\cosh\beta\sin\theta.$$

If we write $\xi=\cosh\alpha\sin\theta-\cosh\beta\cos\theta$,

we get
$$\frac{d\xi}{d\theta}=x,\qquad x^2+\xi^2=\cosh^2\alpha+\cosh^2\beta=1+A,\ \text{say}.$$

Then $$u=\int_{-\cosh\beta}^{+\cosh\alpha}(A-\xi^2)^{-\frac{1}{2}}d\xi,$$
so that
$$\xi_1-\xi_0=\cosh\alpha+\cosh\beta,\quad y_1+y_0=\sinh\alpha+\sinh\beta,$$
taking α and β as positive.

Thus
$$u=2\tan^{-1}[\coth\tfrac{1}{2}(\alpha+\beta)]=\pi-\tan^{-1}[\sinh(\alpha+\beta)].$$

(iii) $u=\int_{-1}^{+1} dx/y. \quad y^2=(1-2hx+h^2)(1-2kx+k^2).$

Then $\quad y_1+y_0=|(1-h)(1-k)|+|(1+h)(1+k)|.$

Thus, if $\qquad |h|<1, \quad |k|<1,$

we have $\qquad y_1+y_0=2(1+hk).$

Hence we find

$$u=\frac{1}{l}\tanh^{-1}\left(\frac{2l}{1+l^2}\right)=\frac{2}{l}\tanh^{-1}(l), \quad \text{if } hk>0;$$

or
$$\frac{1}{l}\tan^{-1}\left(\frac{2l}{1-l^2}\right)=\frac{2}{l}\tan^{-1}(l), \quad \text{if } hk<0,$$

where $l^2=|hk|$.

If $|h|$, or $|k|$, or both, should be greater than 1, the results will be altered; the general form is

$$(2/l)\tanh^{-1}(m) \text{ or } (2/l)\tan^{-1}(m),$$

where m is positive and less than 1, and m^2 is one of the four expressions

$$|hk|, \quad |h^{-1}k|, \quad |hk^{-1}|, \quad |h^{-1}k^{-1}|.$$

This integral occurs in Legendre's method for finding $\int_{-1}^{+1} P_m(x)P_n(x)\,dx$.

(iv) The potential of a uniform rod, lying along the axis of z from $z=-a$ to $z=+a$, is proportional to

$$\int_{-a}^{+a}\frac{d\xi}{\{x^2+y^2+(z-\xi)^2\}^{\frac{1}{2}}}.$$

So, applying our formula, we obtain Tait's result

$$2\tanh^{-1}\left(\frac{2a}{r_1+r_0}\right),$$

where $\quad r_1^2=x^2+y^2+(z-a)^2, \quad r_0^2=x^2+y^2+(z+a)^2.$

The reader will find that in any of these examples the ordinary methods of integration do not lead very readily to results which are as simple as those given above.

3. It is not difficult to obtain a result similar to that of § 2 for the integral

$$u=\int_{x_0}^{x_1}\frac{dx}{ax^2+2bx+c}.$$

In fact, if $ac-b^2$ is positive and equal, say, to γ^2, we have

$$u=\frac{1}{\gamma}\left[\tan^{-1}\left(\frac{ax_1+b}{\gamma}\right)-\tan^{-1}\left(\frac{ax_0+b}{\gamma}\right)\right]$$

$$=\frac{1}{\gamma}\tan^{-1}\left[\frac{\gamma(x_1-x_0)}{ax_1x_0+b(x_1+x_0)+c}\right].$$

The angle in this formula must lie between $-\pi$ and $+\pi$, but a further condition is necessary to fix it precisely; in practice the most satisfactory method is to see if u is positive or negative, which can usually be settled by inspection; the angle is then uniquely determined.

The result can also be put in the form*

$$u=\frac{1}{\gamma}\sin^{-1}\left[\frac{\gamma(x_1-x_0)}{y_1y_0}\right]$$

in case $ax^2+2bx+c$ is positive and equal to y^2. But in this form it is less easy to fix the angle uniquely.

In case $ac-b^2$ is negative and equal to $-k^2$, we have, without ambiguity,

$$u=\frac{1}{k}\tanh^{-1}\left[\frac{k(x_1-x_0)}{ax_1x_0+b(x_1+x_0)+c}\right]=\frac{1}{k}\sinh^{-1}\left[\frac{k(x_1-x_0)}{y_1y_0}\right].$$

When $ac-b^2$ is negative, the roots of $ax^2+2bx+c=0$ are real and may (one or both) fall within the interval (x_0, x_1). If this happens the integral diverges, but its principal value is given by

$$\frac{1}{k}\tanh^{-1}\left[\frac{ax_1x_0+b(x_1+x_0)+c}{k(x_1-x_0)}\right] \text{ or } \frac{1}{k}\tanh^{-1}\left[\frac{k(x_1-x_0)}{ax_1x_0+b(x_1+x_0)+c}\right],$$

according as one or both of the roots are contained between x_0 and x_1.

Finally, in case $ac-b^2=0$, the expression

$$ax^2+2bx+c=(ax+b)^2/a;$$

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* Because $y_1^2y_0^2=[ax_1x_0+b(x_1+x_0)+c]^2+\gamma^2(x_1-x_0)^2$.

and so $$u=\frac{1}{ax_0+b}-\frac{1}{ax_1+b}=\frac{x_1-x_0}{ax_1x_0+b(x_1+x_0)+c},$$

which is equal to the limit of either of the other results when γ (or k) is made to approach zero as a limit.

(i) As a first example, take Hermite's integral

$$\int_0^1 \frac{dx}{x^2-2x\cos\alpha+1}.$$

Applying the formula we get $\frac{1}{2}$ when α is π; and generally

$$\frac{1}{\sin\alpha}\tan^{-1}\left(\frac{\sin\alpha}{1-\cos\alpha}\right)=\frac{1}{2\sin\alpha}(2r\pi+\pi-\alpha),$$

where r is an integer. If we suppose α to lie between 0 and 2π, there is no difficulty in seeing that (since the integral is always positive) we must take $r=0$, so that

$$\int_0^1 \frac{\sin\alpha\, dx}{x^2-2x\cos\alpha+1}=\tfrac{1}{2}(\pi-\alpha),\quad 0<\alpha<2\pi.$$

(ii) Two similar integrals are given by

$$\int_0^1 \frac{dx}{x^2\pm 2x\cosh\alpha+1}.$$

Here we find, taking the $+$ sign

$$\frac{1}{\sinh\alpha}\tanh^{-1}\left(\frac{\sinh\alpha}{1+\cosh\alpha}\right)=\frac{\alpha}{2\sinh\alpha}$$

without ambiguity. With the $-$ sign the principal value is $\frac{1}{\sinh\alpha}\tanh^{-1}\left(\frac{1-\cosh\alpha}{\sinh\alpha}\right)=\frac{-\alpha}{2\sinh\alpha}$. Thus

$$\int_0^1 \frac{\sinh\alpha\, dx}{x^2\pm 2x\cosh\alpha+1}=\pm\tfrac{1}{2}\alpha.$$

(iii) Thirdly, take

$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{a+b\cos\theta+c\sin\theta}=\int_0^1 \frac{2\,dx}{(a-b)x^2+2cx+(a+b)}$$

where $x=\tan\frac{1}{2}\theta$. This is equal to

$$\frac{2}{(a^2-b^2-c^2)^{\frac{1}{2}}}\tan^{-1}\left[\frac{(a^2-b^2-c^2)^{\frac{1}{2}}}{a+b+c}\right].$$

assuming that $a^2 > b^2 + c^2$. The angle will be positive or negative according to the sign of a. Similarly we find

$$\int_0^{\pi} \frac{d\theta}{a + b\cos\theta + c\sin\theta} = \frac{2}{(a^2 - b^2 - c^2)^{\frac{1}{2}}} \tan^{-1}\left[\frac{(a^2 - b^2 - c^2)^{\frac{1}{2}}}{c}\right],$$

the sign of the angle being fixed by a as before. In particular, with $c = 0$, we find (if $a^2 > b^2$)

$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2}{(a^2 - b^2)^{\frac{1}{2}}} \tan^{-1}\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}} = \frac{1}{(a^2 - b^2)^{\frac{1}{2}}} \tan^{-1}\frac{(a^2 - b^2)^{\frac{1}{2}}}{b},$$

$$\int_0^{\pi} \frac{d\theta}{a + b\cos\theta} = \frac{\pm\pi}{(a^2 - b^2)^{\frac{1}{2}}},$$

the sign of the angle being, in both cases,* the same as the sign of a.

(iv) As a fourth example take

$$\int_0^{\infty} \frac{du}{a + b\cosh u} = 2\int_0^1 \frac{dx}{bx^2 + 2ax + b}, \quad (a > b > 0),$$

where $x = e^{-u}$. This gives†

$$\frac{2}{(a^2 - b^2)^{\frac{1}{2}}} \tanh^{-1}\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}} = \frac{1}{(a^2 - b^2)^{\frac{1}{2}}} \tanh^{-1}\frac{(a^2 - b^2)^{\frac{1}{2}}}{a}.$$

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* The determination of the sign is omitted in the ordinary text-books: and usually the extension of the results to cover the case when b is a pure imaginary is incorrectly given. If $b = ic$, we can write

$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{a + ic\cos\theta} = \int_0^{\frac{1}{2}\pi} \frac{a - ic\cos\theta}{a^2 + c^2\cos^2\theta}\, d\theta = \int_0^{\infty} \frac{a\,dt}{(a^2 + c^2) + a^2t^2} - \int_0^1 \frac{ic\,dx}{(a^2 + c^2) - c^2x^2},$$

where $t = \tan\theta$ and $x = \sin\theta$. We then obtain the value

$$\frac{1}{(a^2 + c^2)^{\frac{1}{2}}}\left[\pm\frac{\pi}{2} - i\tanh^{-1}\frac{c}{(a^2 + c^2)^{\frac{1}{2}}}\right].$$

If the integral is taken between the limits 0 and π, the result is

$$\pm\pi / (a^2 + c^2)^{\frac{1}{2}},$$

where the sign is again the same as the sign of a. This case is of importance in expressing Legendre's coefficients as definite integrals.

† If b is a pure imaginary ($= ic$, say) we find by a process similar to that of the last footnote

$$\int_0^1 \frac{a\,dt}{(a^2 + c^2) - a^2t^2} - \int_0^{\infty} \frac{ic\,dx}{(a^2 + c^2) + c^2x^2},$$

where $t = \tanh u$ and $x = \sinh u$. These integrals give the value (c being positive)

$$\frac{1}{(a^2 + c^2)^{\frac{1}{2}}}\left[\tanh^{-1}\frac{a}{(a^2 + c^2)^{\frac{1}{2}}} - \frac{\pi i}{2}\right].$$

4. The two last of the ordinary standard types are

$$u = \int_{x_0}^{x_1} \frac{dx}{(x-p)y}, \qquad v = \int_{x_0}^{x_1} \frac{(Px+Q)\,dx}{z^2 y},$$

where $y^2 = ax^2 + 2bx + c$ and $z^2 = a'x^2 + 2b'x + c'$.

We suppose that* in u *either* $x_1 > x_0 > p$ *or* $p > x_1 > x_0$. It will be found that then

$$u = \frac{2}{q}\tanh^{-1}\left[\frac{(x_1 - x_0)q}{y_0(x_1-p) + y_1(x_0-p)}\right]; \quad \text{if } ap^2 + 2bp + c = q^2;$$

$$\text{or } \frac{2}{r}\tan^{-1}\left[\frac{(x_1 - x_0)r}{y_0(x_1-p) + y_1(x_0-p)}\right], \quad \text{if } ap^2 + 2bp + c = -r^2.$$

We leave the proof to the reader, but it is probably easiest to take $(x-p)^{-1}$ as a new variable and then apply the formula of § 2 to the resulting integral. We could also apply the same transformation as in § 2, which gives

$$x = \frac{x_0 + 2y_0 t + (ax_0 + 2b)t^2}{1 - at^2}.$$

As an example, take

$$\int_0^\infty \frac{du}{a + b\cosh u} = \int_1^\infty \frac{dx}{(a+bx)(x^2-1)^{\frac{1}{2}}}, \quad (a > b > 0),$$

where $x = \cosh u$. This gives

$$\frac{2}{(a^2-b^2)^{\frac{1}{2}}}\tanh^{-1}\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}},$$

as found in (iv) of § 3.

The integral v can be expressed as the sum of two integrals of the type u, when the factors of the quadratic $a'x^2 + 2b'x + c'$ are real; thus the only point of novelty occurs when these factors are *complex*. It is known† that then real numbers p, q,

* If p is between x_0 and x_1, the integral is divergent, but its principal value is easily found to be

$$\frac{2}{q}\tanh^{-1}\left[\frac{y_0(x_1-p) + y_1(x_0-p)}{(x_1-x_0)q}\right].$$

† See, for example, my forthcoming tract on *Quadratic Forms* (No. 3 of the Cambridge University Press Tracts).

The fact that A, B must have the same sign is evident because the roots of $= 0$ are complex. This sign may be supposed positive without loss of generality.

A, B, λ, μ can be found such that

$$y^2 = A\lambda\,(x-p)^2 + B\mu\,(x-q)^2,$$

$$z^2 = \;\; A\,(x-p)^2 + B\,(x-q)^2,$$

and A, B will be positive.

Thus the integral v can be divided into two others

$$v_1 = \int_{x_0}^{x_1} \frac{(x-p)\,dx}{yz^2}, \quad v_2 = \int_{x_0}^{x_1} \frac{(x-q)\,dx}{yz^2},$$

so that $$v = Lv_1 + Mv_2,$$

where L, M are constants and are easily expressed in terms of P, Q and p, q. If $(x-q)/z$ is introduced as a new independent variable in v_1 (and $(x-p)/z$ in v_2) it will be found that the formula of § 2 gives

$$A\,(p-q)\,v_1 = -\frac{2}{[B\,(\lambda-\mu)]^{\frac{1}{2}}}\tan^{-1}\frac{[B\,(\lambda-\mu)]^{\frac{1}{2}}\,[(x_1-q)\,z_0-(x_0-q)\,z_1]}{y_1z_0+y_0z_1},$$

$$B\,(p-q)\,v_2 = \;\; \frac{2}{[A\,(\lambda-\mu)]^{\frac{1}{2}}}\tanh^{-1}\frac{[A(\lambda-\mu)]^{\frac{1}{2}}\,[(x_1-p)z_0-(x_0-p)z_1]}{y_1z_0+y_0z_1},$$

assuming* that λ is greater than μ.

The special case given by $b=0$, $b'=0$ has some interest; it may be regarded as the limit of the last case when $p=0$, and q tends to ∞, while B tends to zero and Bq^2 to c'. Hence, or by a direct calculation, we find

$$\int\frac{dx}{yz^2} = \frac{2}{k\gamma}\tanh^{-1}\frac{k\,(x_1z_0-x_0z_1)}{\gamma\,(y_1z_0+y_0z_1)}, \quad \int\frac{xdx}{yz^2} = \frac{2}{k\alpha}\tan^{-1}\frac{k\,(z_1-z_0)}{\alpha\,(y_1z_0+y_0z_1)},$$

where $$y^2 = ax^2+c, \quad z^2 = a'x^2+c',$$

$$k^2 = ac'-a'c, \quad \alpha^2 = a', \quad \gamma^2 = c'.$$

If $ac'-a'c$ is negative $(=-l^2)$, k must be replaced by l and the circular and hyperbolic functions interchanged.

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* The case $\lambda=\mu$ can only occur when y^2 is a mere multiple of z^2, because the quadratic form z^2 is necessarily positive. The indefinite integral is then algebraic, of the form $(Rx+S)/z$.

NOTE ON THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS BY MEANS OF DEFINITE INTEGRALS.

By *H. Bateman*.

In the standard methods for solving linear differential equations by means of definite integrals it is assumed that the integral which represents a solution of the equation can be differentiated under the integral sign. Now it is possible to satisfy a linear differential equation by means of an integral which cannot be differentiated by the rule of Leibnitz, but which can be integrated.

The method to be adopted is as follows: Let $P_x(w)=0$ be a linear differential equation of the nth order, and suppose that two functions $f(x, t)$, $F(x, t)$ and a second differential equation $Q_t(v)=0$ have been found such that

$$P_x(f) \equiv Q_t(F).$$

If T is an integrating factor of the equation $Q_t(v)=0$, the integral which we consider is

$$w=\int_c f(x, t)\, T dt.$$

Assuming first that this can be differentiated n times, we find that

$$P_x(w)=\int_c P_x(f)\, T dt=\int_c Q_t(F)\, T dt.$$

Now $Q_t(F)T$ is a perfect differential dV. Accordingly, if the path of integration is chosen so that V takes the same value at each end, we shall have $P_x(w)=0$, and one solution of the equation is known.

Next, suppose that the integral can only be differentiated m times, but on the other hand can be integrated $n-m$ times. We then integrate the identity

$$P_x(f)=Q_t(F)$$

$n-m$ times, and transform each term by integrating by parts so as to remove as many of the high differential coefficients as is necessary. We shall finally obtain an equation of the form

$$R_x(f)=Q_t\{\iiint F dx dx\}=Q_t(\phi),$$

where R_x is an operator consisting partly of integrations and partly of differentiations, the integrations will in general not act on f alone, but on f multiplied by some function of x.

Now operate on the integral $\int_c f(x, t)\, T dt$, assuming that all the operations can be effected, then

$$R_x(w) = \int_c R_x(f)\, T dt = \int_c Q_t(\phi)\, T dt = \int_c dW.$$

The boundary condition which we have to satisfy is now slightly different; assuming that a suitable path has been chosen we shall have

$$R_x(w) = 0.$$

Differentiating this $n-m$ times so as to get rid of the integrations, we return to the original equation $P_x(w) = 0$. Accordingly, the quantity w is a solution.

By way of illustration I shall consider the equation

$$x\frac{d^2w}{dx^2} + \frac{dw}{dx} + xw = 0,$$

which is satisfied by the functions $J_0(x)$ and $K_0(x)$. It can easily be verified that*

$$\left(x\frac{d^2}{dx^2} + \frac{d}{dx} + x\right) \sin(x \cosh t) = \frac{d}{dt}\{\sinh t \cos(x \cosh t)\}.$$

Now it is by no means easy to choose a real path of integration which will satisfy the boundary condition

$$\{\sinh t \cos(x \cosh t)\}_1{}^2 = 0,$$

and at the same time allow the integral $\int \sin(x \cosh t)\, dt$ to be differentiated,† but if we integrate the above equation twice, obtaining the identity

$$x \sin(x \cosh t) - \int \sin(x \cosh t)\, dx + \iint x \sin(x \cosh t)\, dx$$
$$= -\frac{d}{dt}\left\{\frac{\sinh t}{\cosh^2 t} \cos(x \cosh t)\right\},$$

it is easy to see that the boundary condition is satisfied by taking the limits $t = 0$ and $t = \infty$. Thus we obtain the solution

$$w = \int_0^\infty \sin(x \cosh t)\, dt,$$

which can be identified‡ with $\frac{\pi}{2} J_0(x)$.

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* This result is obtained at once from the general formula for the Laplace transformation. See Forsyth's *Differential Equations*, p. 224.

† The conditions can be satisfied by taking the limits to be 0 and $i\pi$.

‡ Whittaker's *Analysis*, Ex. XII., p. 306. The above integral is indeterminate in the vicinity of $x = 0$, but so long as we do not integrate through this point our method will be legitimate.

THE POWER SERIES FOR $\sin x$, $\cos x$.

By *Prof. E. J. Nanson.*

THE power series for $\sin x$, $\cos x$ follow at once from the two inequalities

$$\sin x \text{ lies between } S_n,\ S_{n+1} \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\cos x \text{ lies between } C_n,\ C_{n+1} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where
$$S_n = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!},$$

$$C_n = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^{n-1} x^{2n-2}}{(2n-2)!},$$

and, as is well known, these two inequalities are readily derived from the rules for differentiating $\sin x$, $\cos x$. But as Professor Hill has recently (*Messenger*, Vol. XXXV., pp. 58–69) called attention to the importance of simplifying the trigonometrical proofs of (1), (2), and has shown how to extend Le Cointe's proof of (1) for the case $n = 1$ so as to obtain a general proof of (1), (2), so long as x is acute it may not be amiss to show how the older and simpler method of proving (1) for the case $n = 1$ by making use of Euler's product can also be extended to prove (1), (2) for all angles less than four right angles.

In the first place it is readily shown that (1) follows from (2) for all real values of x. For if there are m angles $\alpha, \beta, \gamma, \ldots$, we have

$$\cos\alpha\cos\beta\cos\gamma\ldots = 2^{-m}\Sigma\cos(\pm\alpha\pm\beta\pm\gamma\pm\ldots).$$

Hence, according as

$$\cos x \gtrless a_0 + \Sigma a_r x^{2r},$$

we have
$$\cos\alpha\cos\beta\cos\gamma\ldots \gtrless a_0 + \Sigma a_r S_r,$$

where
$$S_r = 2^{-m}\Sigma(\pm\alpha\pm\beta\pm\gamma\pm\ldots)^{2r}.$$

Now take $\alpha, \beta, \gamma, \ldots$ to be $\frac{x}{2}, \frac{x}{2^2}, \frac{x}{2^3}, \ldots, \frac{x}{2^m}$. Then we have

$$S_r = \frac{2x^{2r}}{2^{r(2r+1)}} t_r,$$

where t_r is the sum of the $2r$th powers of the odd numbers from 1 to $2^m - 1$, so that

$$t_r = \frac{1}{2r+1}\,\frac{2^{m(2r+1)}}{2}(1+\epsilon),$$

where ϵ can be made as small as we please by taking m large enough.

Thus we have

$$S_r = \frac{x^{2r}}{2r+1}(1+\epsilon).$$

Again, since

$$\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\ldots\cos\frac{x}{2^n} = \frac{\sin x}{2^n \sin\frac{x}{2^n}},$$

it follows that

$$\frac{\sin x}{2^n \sin\frac{x}{2^n}} \gtrless a_0 + \Sigma \frac{1}{2r+1} a_r (1+\epsilon) x^{2r},$$

according as

$$\cos x \gtrless a_0 + \Sigma a_r x^{2r}.$$

Now take

$$a_r = \frac{(-1)^r}{(2r)!},\ r = 0,\ 1,\ 2,\ \ldots,\ n-1,$$

and make m infinite, and we see that (1) follows from (2).

In the next place, as shown by Professor Hill, if (1) is granted it follows that $\cos x$ lies between C_{n+1}, C_{n+2} provided x is acute. But as a matter of fact the proof applies if x lies between zero and four right angles. To show this, let

$$U_n = u_1 - u_2 + u_3 - \ldots + (-1)^{n-1} u_n,$$

with similar formulæ for V, v, W, w, and let

$$w_r = u_1 v_r + u_2 v_{r-1} + \ldots + u_r v_1,$$

then it is readily seen that

$$(-1)^n (U_n V_n - W_n)$$
$$= u_2 v_n + u_3 (v_{n-1} - v_n) + u_4 (v_{n-2} - v_{n-1} + v_n) + \ldots$$
$$+ u_n \{v_2 - v_3 + v_4 - \ldots + (-1)^n v_n\}.$$

Hence it follows that, if, for $r > 1$, u_r is positive and $v_r > v_{r+1}$, or v_r is positive and $u_r > u_{r+1}$, then $U_n V_n - W_n$ is positive or negative according as n is even or odd. Now if u_r, v_r are both positive it follows that $U_{n+1} \gtrless U_n$, $V_{n+1} \gtrless V_n$, and therefore also $U_{n+1} V_{n+1} < U_n V_n$ according as n is even or odd. Hence

$$W_n, \ U_n V_n, \ U_{n+1} V_{n+1}, \ W_{n+1}$$

are in order of magnitude. Thus if U lies between U_n, U_{n+1}, and V between V_n, V_{n+1}, it follows that UV lies between W_n, W_{n+1} provided that for $r > 1$ u_r, v_r are positive, and either $u_r > u_{r+1}$ or $v_r > v_{r+1}$.

Taking $U_n = V_n = S_n$, it follow from (1) that $\sin^2 x$ lies between W_n, W_{n+1}, where

$$W_n = w_1 - w_2 + w_3 - \ldots + (-1)^{n-1} w_n,$$

$$w_r = x^{2r} \Sigma \frac{1}{(2p-1)!\,(2q-1)!},$$

Σ denoting summation for

$$p + q = r + 1, \quad p = 1, 2, \ldots, r,$$

so that
$$w_r = \frac{2^{2r-1} x^{2r}}{(2r)!}.$$

Hence, using the formula $\cos 2x = 1 - 2\sin^2 x$ and replacing $2x$ by x, it follows from (1) that $\cos x$ lies between C_{n+1}, C_{n+2}.

For the applicability of the process it is necessary that $x^2 < 20$, or with the final and altered value of x that $x^2 < 80$ or $x < 8 \cdot 9$, and this is satisfied if $x < 2\pi$. Since, then, it is known that (1), (2) are true when $n = 1$ for all real values of x, it follows by induction that (1) and (2) are true when $0 < x < 2\pi$.

Melbourne,
October 24th, 1905.

A FORMULA FOR THE PRIME FACTORS OF ANY NUMBER.

By *G. H. Hardy*, Trinity College, Cambridge.

§ 1. My object in this note is to define a function $\theta(x)$ of x such that, if x is a positive integer n, $\theta(x)$ is the largest prime contained in x.

Let us suppose for a moment that $\theta(x)$ is such a function. Then if $x = a^\alpha b^\beta c^\gamma \ldots$, a, b, ... being primes arranged in descending order, and α, β, ... positive integers,

$$\theta(x) = a.$$

Hence $$x/\theta(x) = a^{\alpha-1} b^\beta c^\gamma \ldots,$$

and the function

$$\theta_2(x) = \theta\{x/\theta(x)\}$$

is equal to a or b according as $\alpha > 1$ or $\alpha = 1$; and it is clear that if we define a series of functions

$$\theta(x), \;\; \theta_2(x), \;\; \theta_3(x)$$

by the equations

$$\theta_2(x) = \theta\{x/\theta(x)\}, \quad \theta_3(x) = \theta\{x/\theta_2(x)\}, \ldots,$$

we obtain, on replacing each function by its value, the whole series of prime factors of x, each factor occurring the same number of times in the series as in x. If, for example,

$$x = ab^3c^2d^2,$$

$$\theta(x) = a, \;\; \theta_2(x) = \theta_3(x) = \theta_4(x) = b,$$

$$\theta_5(x) = \theta_6(x) = c, \;\; \theta_7(x) = \theta_8(x) = d,$$

while $\theta_9(x)$ and all later members of the series of functions are equal to unity.

The function $\theta(x)$ is, in fact, defined by the equation

$$\theta(x) = \lim_{r=\infty} \; \lim_{m=\infty} \; \lim_{n=\infty} \; \sum_{\nu=0}^{m} \left[1 - \left(\cos\left\{(\nu!)^r \pi/x\right\}\right)^{2n}\right],$$

the passages to the limit being performed in the order from right to left.

I need hardly say that this result does not pretend to be more than a curiosity, and that its value for any kind of

application is *nil.* At the same time it seems worth while to point out that it is possible to write down a closed analytical expression which properly represents a *function* of x in Euler's sense, and which does possess the at first sight astonishing property stated above; and I am not aware that a similar formula has been given before. I may also remark that the form of $\theta(x)$ was suggested to me by the expression

$$y = \lim_{m=\infty} . \lim_{n=\infty} . (\cos m!\pi x)^{2n},$$

given by Pringsheim and Lebesgue for the function y which is unity for all rational, and zero for all irrational values of x.

§ 2. Let us suppose that

$$x = a^{\alpha} b^{\beta} c^{\gamma} \ldots,$$

so that a is the largest prime contained in x. Then $\nu!$ is divisible by $abc\ldots$ if, and only if, $\nu \geqq a$. And, for such values of ν, $(\nu!)^r$ is divisible by x provided r is greater than or equal to the greatest of $\alpha, \beta, \gamma, \ldots$. Hence, if we give r a sufficiently great but fixed value,

$$\cos^2\{(\nu!)^r\pi/x\} < 1, \quad (\nu = 0, 1, \ldots, a-1),$$

$$\cos^2\{(\nu!)^r\pi/x\} = 1, \quad (\nu \geqq a).$$

And therefore, if $m \geqq a$,

$$\lim_{n=\infty} . \sum_{\nu=0}^{m} \left[1 - (\cos\{(\nu!)^r\pi/x\})^{2n}\right] = a.$$

This equation holds for all sufficiently large values of r and m, whatever be x. Hence in all cases

$$\theta(x) = a.$$

It is easy to see that the order in which m and r are made to tend to infinity is immaterial—or they may be made to tend to infinity simultaneously: but it is essential that the passage to $n = \infty$ should take precedence of either.

The value of $\theta(x)$ for any rational but not integral value of x is easily found, but is of no particular interest; for all irrational values of x, $\theta(x)$ has the 'improper value' $+\infty$.

ON THE REVERSION OF AN ASYMPTOTIC EXPANSION.

By *E. Cunningham*, St. John's College, Cambridge.

Let y be a branch of a function of x possessing the asymptotic expansion

$$x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots,$$

so that within a certain angle A at $x = 0$, and within the circle $|x| = \rho$, and for all values of n,

$$|y - (x + \ldots + a_nx^n)| < \epsilon|x|^n,$$

it being possible to find a non-zero quantity ρ, however small ϵ may be.

Suppose now the coefficients $b_1, b_2, \ldots$ are determined, so that if the series $y + b_2y^2 + b_3y^3 + \ldots$ is substituted in the expression

$$y - x - a_2x^2 - \ldots,$$

the successive coefficients of $y, y^2, \ldots$ vanish identically. Then the series $y + b_2y^2 + \ldots$ cannot converge within a finite circle about $y = 0$, for if it did the original series $x + a_2x^2 + \ldots$ would also converge within a finite circle about $x = 0$. But it is to be proved that the former series is an asymptotic expansion of that branch of x considered as a function of y, which vanishes with y, valid within an angle enclosed in A, but differing from A by an arbitrary small angle.

The asymptotic expansion of y shows, by putting $n = 1$, that $|y - x| < \epsilon|x|$ within the area common to A, and the circle $|x| = \rho$, so that y lies within the angle B obtained by diminishing A by ϵ on either side.

Conversely, if y lies within B, x lies within A.

Also $$|x| - |y - x| > (1 - \epsilon)|x|,$$

so that $$|y| > |x|(1 - \epsilon),$$

or $$|x| < \frac{|y|}{1 - \epsilon},$$

Consequently, if $$|y| < \rho(1 - \epsilon),$$

$$|x| < \rho.$$

Now let $$y - x - a_2x^2 \ldots - a_nx^n = y_n.$$

Then y_n is a function of x, which for $|x| < \rho$, is less in modulus than $\epsilon |x|^n$. If then y_n be considered as a function of y, we have, provided $|y| < \rho(1-\epsilon)$ and y lies within B,

$$|y_n| < \epsilon |x|^n < \frac{\epsilon |y|^n}{(1-\epsilon)^n} < \eta |y^n|,$$

where η stands for $\dfrac{\epsilon}{(1-\epsilon)^n}$, and can, by making ϵ small enough, be made arbitrarily small.

Now consider the expression

$$x - y - b_2 y^2 \ldots - b_n y^n = x_n,$$

the coefficients b_2, $b_3 \ldots$ being derived as above.

Then in the equation

$$y_n = y - (y + b_2 y^2 + \ldots + b_n y^n + x_n) \ldots - a_n (y + b_2 y^2 + \ldots + b_n y^n + x_n)^n$$

the parts of the coefficients of y, y^2, ..., y^n which do not contain x_n vanish identically, and the equation reduces to

$$y_n = x_n p_{n(n-1)}(y) + x_n^2 p_{n(n-2)}(y) + \ldots - a_n x_n^n + Q(y),$$

where p_r represents a polynomial in y of degree r and Q is polynomial in y of the form $\sum\limits_{n+1}^{n^2} q_r y^r$. In solving this equation for x_n that branch of x_n is to be found which vanishes with y. That there is only one such branch is clear, since for $y = 0$ the equation reduces to

$$0 = x_n + a_1 x_n^2 + \ldots + a_n x^n,$$

of which not more than one root is zero.

This branch may be developed in powers of $\dfrac{y_n - Q(y)}{p_{n(n-1)}(y)}$, the coefficients being polynomials in y, provided y is sufficiently small and within B, and therefore putting $y_n = y^n Y_n$,

$$x_n = -y^n Y_n + y^{n+1} \Omega,$$

where Ω is a series of powers of Y_n and y, converging when y lies within a certain finite circle and also within the angle B. But for $|y| < \rho(1-\epsilon)$, $|Y_n| < \eta$ and Ω is finite.

Thus
$$\left|\frac{x_n}{y^n}\right| \leqq |Y_n| + |y|\,|\Omega|$$

$$< \eta + \Omega\rho(1-\epsilon),$$

Ω being finite, and this last expression can, by taking ρ

sufficienly small, be made less than any assigned small quantity.

Thus the series

$$y + b_2 y^2 + b_3 y^3 + \ldots + b_n y^n$$

is an asymptotic expansion of one branch of x considered as a function of y within an angle B, which differs from A by an *arbitrarily small* amount.

DIRECT DEFINITION OF AN n^{th} DIFFERENTIAL COEFFICIENT.

By *R. Hargreaves, M.A.*

If $f(z)$ is a function of z which can be expanded by Taylor's theorem within a field in which the n values $\theta_1 \ldots \theta_n$ are comprised, and if θ is the arithmetic mean of these values, then

$$\frac{f(\theta_1)}{(\theta_1 - \theta_2)\ldots(\theta_1 - \theta_n)} + \frac{f(\theta_2)}{(\theta_2 - \theta_1)\ldots(\theta_2 - \theta_n)} + \ldots$$

$$= \frac{1}{(n-1)!} \frac{d^{n-1} f(\theta)}{d\theta^{n-1}} + \frac{\Sigma(\theta - \theta_r)^2}{2(n+1)!} \frac{d^{n+1} f(\theta)}{d\theta^{n+1}}$$

$$+ \frac{\Sigma(\theta - \theta_r)^3}{3(n+2)!} \frac{d^{n+2} f(\theta)}{d\theta^{n+2}} + \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

the following terms being less simple. Thus if $\theta_1 \ldots \theta_n$ all approach the mean θ, the limit of the sum gives a definition of the $(n-1)^{th}$ differential coefficient without reference to earlier coefficients; and, moreover, the residue, as a limit is approached, is a sum of squares of small quantities.

$$\left.\begin{array}{ll} \text{Write} & \theta_1 = \theta + x_1, \quad \theta_2 = \theta + x_2, \ \ldots \\ \text{so that} & \theta_1 - \theta_2 = x_1 - x_2, \ \ldots \text{ and } \Sigma x_r = 0 \end{array}\right\} \ldots\ldots\ldots\ldots(2).$$

If $f(\theta_1)$, ... are expanded by Taylor's theorem, the series on the left-hand of (1) is

$$f(\theta)\,\Sigma \frac{1}{(x_1 - x_2)\ldots(x_1 - x_n)} + f'(\theta)\,\Sigma \frac{x_1}{(x_1 - x_2)\ldots(x_1 - x_n)}$$

$$+ \frac{f''(\theta)}{2!}\,\Sigma \frac{x_1^2}{(x_1 - x_2)\ldots(x_1 - x_n)} + \ldots.$$

The sum $\Sigma \dfrac{x_1^m}{(x_1 - x_2)\ldots(x_1 - x_n)} = 0$ if $m < (n-1)$, and if $m = n-1,\ n, n+1, \ldots$ its values are $1,\ 0,\ -a_2,\ -a_3,\ -a_4 + a_2^2, \ldots$, where $(x - x_1)\ldots(x - x_n) = x^n + a_2 x^{n-2} + \ldots + a_n$; or if we prefer expression by powers only, they are

$$1,\ 0,\ \frac{S_2}{2},\ \frac{S_3}{3},\ \frac{S_4}{4} + \frac{S_2^2}{8},\ \ldots,$$

where $S_2 = \Sigma x_r^2 = \Sigma(\theta - \theta_r)^2$. The use of these well-known algebraical results therefore establishes (1).

Now use Cauchy's formula $f(\theta_1) = \dfrac{1}{2\pi\iota}\displaystyle\int \dfrac{f(z)\,dz}{z - \theta_1}$ in conjunction with (1). The left-hand member

$$= -\frac{1}{2\pi\iota}\int f(z)\,dz\left[\frac{1}{(\theta_1 - z)(\theta_1 - \theta_2)\ldots(\theta_1 - \theta_n)} + \ldots\right]$$

$$= \frac{1}{2\pi\iota}\int \frac{f(z)\,dz}{(z - \theta_1)\ldots(z - \theta_n)},$$

in virtue of the summation formula with $m = 0$ and an extra letter z. The last integral is then equal to the right-hand member of (1), and if we proceed to the limit

$$\frac{1}{2\pi\iota}\int \frac{f(z)\,dz}{(z - \theta)^n} = \frac{1}{(n-1)!}\,\frac{d^{n-1}f(\theta)}{d\theta^{n-1}} \quad \ldots\ldots\ldots\ldots (3),$$

Again, in (1), suppose θ_2 to coalesce with θ_1, the left-hand member becomes

$$\frac{d}{d\theta_1}\,\frac{f(\theta_1)}{(\theta_1 - \theta_3)\ldots(\theta_1 - \theta_n)} + \frac{f(\theta_3)}{(\theta_3 - \theta_1)^2(\theta_3 - \theta_4)\ldots(\theta_3 - \theta_n)}\ldots;$$

if three coalesce it becomes

$$\frac{1}{2}\,\frac{d^2}{d\theta_1^2}\,\frac{f(\theta_1)}{(\theta_1 - \theta_4)\ldots(\theta_1 - \theta_n)} + \frac{f(\theta_4)}{(\theta_4 - \theta_1)^3(\theta_4 - \theta_5)\ldots(\theta_4 - \theta_n)}\ldots,$$

and so on. These results follow by application of the limiting form of (1) to the coalescing section. On the right-hand of (1) we count θ_1 twice, three times, &c., in taking the mean θ and in forming the sums $\Sigma(\theta - \theta_r)^2, \ldots$. The continuation of the process leads ultimately to the identity

$$\frac{1}{(n-1)!}\,\frac{d^{n-1}f(\theta)}{d\theta^{n-1}} = \frac{1}{(n-1)!}\,\frac{d^{n-1}f(\theta)}{d\theta^{n-1}}.$$

A PROOF OF THE MULTIPLICATION THEOREM FOR DETERMINANTS.

By *R. F. Muirhead, D.Sc.*

$A \equiv$ the determinant of nth order of which a_{pq} is the element common to the pth row and qth column.

$B \equiv$ ditto for b_{qp}.

$\Delta \equiv$ ditto for $\overset{r}{\Sigma} a_{pr} b_{qr}$, where $\overset{r}{\Sigma}$ denotes summation from $r = 1$ to $r = n$.

It is required to prove $\Delta = A \,.\, B$.

We shall first prove the lemma $\Delta/\Delta_0 = A/A_0$, where A_0, Δ_0 denote the values of A, Δ respectively when all the elements of the pth row of A vanish, except a_{pq}; so that $A_0 = a_{pq}A_{pq}$, where A_{pq} is the minor of a_{pq} in A.

Let us multiply the elements of the 1st, 2nd, ..., nth rows of Δ by A_{1q}, A_{2q}, A_{3q}, ..., A_{nq} respectively, and add the results by columns. Thus the result in the mth column is $\overset{s}{\Sigma} A_{sq} \overset{r}{\Sigma} a_{sr} b_{mr}$, *i.e.* $\overset{r}{\Sigma}\{b_{mr} \overset{s}{\Sigma} a_{sr} A_{sq}\}$, which, since $\overset{s}{\Sigma} a_{sr} A_{sq} = A$ or 0 according as $r = q$ or $r \neq q$, reduces to $b_{mq}A$, or $a_{pq} b_{mq} \cdot \dfrac{A}{a_{pq}}$.

Let us now form a determinant which differs from Δ only in its pth row, the new row consisting of the sums just formed. The new determinant therefore $= A_{pq} \,.\, \Delta$,* and after dividing by the common factor $\dfrac{A}{a_{pq}}$ of the pth row elements, it clearly reduces to Δ_0, so that we have

$$A_{pq} \,.\, \Delta = \frac{A}{a_{pq}} \Delta_0,$$

or, since $A_{pq} \,.\, a_{pq} = A_0$, $\Delta/\Delta_0 = A/A_0$.

This proves the Lemma.

Now let $A_1 \equiv$ the value of A_0, when $p = 1$, $q = 1$,

i.e. „ $A_1 \equiv$ „ „ A,

when all elements of its 1st row vanish, except a_{11},

and let $A_2 \equiv$ the value of A_1,

when all elements of its 2nd row vanish, except a_{22},

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* Because the new row is got by multiplying each of its old elements by A_{pq} and adding thereto equi-multiples of corresponding elements of other rows.

and let $A_3 \equiv$ the value of A_2,

when all elements of its 3rd row vanish, except a_{33},

....

and let $A_n \equiv$ the value of A_{n-1},

when all elements of the nth row vanish, except a_{nn},

And let suffixes attached to Δ affect it in the same way. Then, by the Lemma, we have

$$\frac{\Delta}{\Delta_1} = \frac{A}{A_1}, \quad \frac{\Delta_1}{\Delta_2} = \frac{A_1}{A_2}, \quad \ldots, \quad \frac{\Delta_{n-1}}{\Delta_n} = \frac{A_{n-1}}{A_n}.$$

Hence
$$\frac{\Delta}{\Delta_n} = \frac{A}{A_n}.$$

Now
$$A_n = a_{11} a_{22} a_{33} \ldots a_{nn},$$

and
$$\Delta_n = \begin{vmatrix} a_{11}b_{11}, & a_{11}b_{21}, & a_{11}b_{31}, & \ldots, & a_{11}b_{n1} \\ a_{22}b_{12}, & a_{22}b_{22}, & a_{22}b_{32}, & \ldots, & a_{22}b_{n2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{nn}b_{1n}, & \cdots & \cdots & \cdots & a_{nn}b_{nn} \end{vmatrix}$$

$$= a_{11} a_{22} a_{33} \ldots a_{nn} . B.$$

Hence
$$\Delta = A . B. \qquad \text{Q. E. D.}$$

November, 1905.

ON AN EXTENSION OF THE FOURIER METHOD OF EXPANDING A FUNCTION IN A SERIES OF SINES AND COSINES.

By *Charles H. Lees, D.Sc.*

It is sometimes necessary in mathematical physics to express a quantity y, which is a function $f(x)$ of x, in a Fourier series of the form

$$f(x) \equiv y = b_0 + b_1 \cos \beta_1 x + b_2 \cos \beta_2 x + b_3 \cos \beta_3 x + \ldots$$
$$+ a_1 \sin \alpha_1 x + a_2 \sin \alpha_2 x + a_3 \sin \alpha_3 x + \ldots,$$

where the α's and β's are such that some further condition is satisfied by y for a particular value c of x.

Thus, in the cooling of a sphere of radius c in a gas at zero temperature, if x is the radius of a concentric shell and y is x times the temperature of it, we have, since the sine series alone is necessary,

$$\alpha \cos \alpha x + p \sin \alpha x = 0 \text{ when } x = c,$$

p being a constant. This equation determines the values of the coefficients $\alpha_1, \alpha_2, \alpha_3, \ldots$ and the coefficients $a_1, a_2, a_3, \ldots$ are then determined as in Fourier's *Théorie Analytique de la Chaleur*, Chap. V., Art. 291.

A similar method of procedure may be adopted if the series is one of cosines only, and the condition to be satisfied is

$$\alpha \sin \alpha x + p \cos \alpha x = 0 \text{ when } x = c.$$

In the *Messenger* for September, 1903 (Vol. XXXIII., p. 70), Mr. Andrew Stephenson has shown how the coefficients a_1, a_2, a_3, &c. may be calculated in case the series is one of sines only, and the condition

$$\cos \alpha x + p\alpha \sin \alpha x = q \frac{\sin \alpha c}{\alpha}$$

is satisfied when $x = c$, and how the coefficients b_0, b_1, ... may be calculated in case the series is one of cosines only, and the condition

$$\sin \alpha x + p\alpha \cos \alpha x = q \frac{\cos \alpha c}{\alpha}$$

is satisfied when $x = c$.

I propose to show that these are special cases of a rather more general rule in which the calculation of the coefficients is possible.

Let us assume that a function $y \equiv f(x)$ can be expanded in a convergent series of sines and cosines of multiples of x as follows:

$$\left.\begin{aligned} y = f(x) = b_0 + b_1 \cos \beta_1 x + b_2 \cos \beta_2 x + b_3 \cos \beta_3 x + \ldots \\ + a_1 \sin \alpha_1 x + a_2 \sin \alpha_2 x + a_3 \sin \alpha_3 x + \ldots \end{aligned}\right\} \ldots\ldots(1),$$

where the coefficients, α and β, are such that

$$\phi(D)\, y = C \text{ when } x = c \ldots\ldots\ldots\ldots\ldots(2),$$

where C is independent of x, D is written for $\frac{d}{dx}$, $\phi(D)$ is a polynomial of the form

$$\phi(D) \equiv \ldots A_2 D^{-2} + A_1 D^{-1} + A_0 + a_1 D + a_2 D^2 + \ldots,$$

and the constants introduced by the integrations of the type $A_n D^{-n}$ are included in A_0.

To find the values of the coefficients a, b, α, β.

Dividing $\phi(D)$ into its even and odd parts, we write

$$\phi(D) \equiv \psi(D^2) + D\chi(D^2).$$

Now
$$\begin{aligned}
\psi(D^2)\cos\beta x &= \psi(-\beta^2)\cos\beta x,\\
\psi(D^2)\sin\beta x &= \psi(-\beta^2)\sin\beta x,\\
D\chi(D^2)\cos\beta x &= -\beta\chi(-\beta^2)\sin\beta x,\\
D\chi(D^2)\sin\beta x &= \beta\chi(-\beta^2)\cos\beta x.
\end{aligned}$$

Hence the equation $\phi(D)y = C$, for $x = c$ becomes on substitution,

$$\begin{aligned}
&\left(A_0 + A_1 c + A_2\frac{c^2}{2} + \dots\right)b_0\\
&\quad + \Sigma b_n\{\psi(-\beta_n^2)\cos\beta_n c - \beta_n\chi(-\beta_n^2)\sin\beta_n c\}\\
&\quad + \Sigma a_n\{\psi(-\alpha_n^2)\sin\alpha_n c + \alpha_n\chi(-\alpha_n^2)\cos\alpha_n c\} = C.
\end{aligned}$$

The general condition (2) can therefore only be satisfied if

$$\left.\begin{aligned}
&b_0 = C\Big/\left(A_0 + A_1 c + A_2\frac{c^2}{2} + \dots\right)\\
&\psi(-\beta_n^2)\cos\beta_n c - \beta_n\chi(-\beta_n^2)\sin\beta_n c = 0\\
&\psi(-\alpha_n^2)\sin\alpha_n c + \alpha_n\chi(-\alpha_n^2)\cos\alpha_n c = 0
\end{aligned}\right\}.$$

The last two equations give

$$\left.\begin{aligned}
\tan\beta_n c &= \frac{\psi(-\beta_n^2)}{\beta_n\chi(-\beta_n^2)}\\
\cot\alpha_n c &= -\frac{\psi(-\alpha_n^2)}{\alpha_n\chi(-\alpha_n^2)}
\end{aligned}\right\}\dots\dots\dots\dots\dots(3)$$

transcendental equations which determine the coefficients β, α of x in the cosine and sine terms of the expansion.

To determine the coefficients a, b, we now for convenience divide $y \equiv f(x)$ into its even and odd parts $f_0(x)$ and $f_1(x)$ and write

$$\begin{aligned}
f_0(x) &\equiv b_0 + b_1\cos\beta_1 x + b_2\cos\beta_2 x + b_3\cos\beta_3 x + \dots,\\
f_1(x) &\equiv \qquad a_1\sin\alpha_1 x + a_2\sin\alpha_2 x + a_3\sin\alpha_3 x + \dots.
\end{aligned}$$

Multiply both sides of the first of these equations by $\cos\beta_m x$, where β_m is one of the coefficients of x determined from (3), and integrate between the limit 0 and c. We have

$$\int_0^c f_0(x)\cos\beta_m x dx = \int_0^c b_0 \cos\beta_m x dx + b_m\int_0^c \cos^2\beta_m x dx$$
$$+\sum_{k\neq m} b_k \int_0^c \cos\beta_k x\cos\beta_m x dx,$$

where the summation in the last member is over all values of k exclusive of m,

$$i.e. \int_0^c \{f_0(x)-b_0\}\cos\beta_m x dx = b_m\frac{c}{2}\left\{1+\frac{\sin 2\beta_m c}{2\beta_m c}\right\}$$
$$+\sum_{k\neq m} b_k \frac{1}{\beta_k^2-\beta_m^2}\{\beta_k\sin\beta_k c\cos\beta_m c-\beta_m\cos\beta_k c\sin\beta_m c\}.$$

Putting in the last term $\beta\sin\beta c = \dfrac{\psi(-\beta^2)}{\chi(-\beta^2)}\cos\beta c$ in virtue of (3), this equation becomes

$$\int_0^c \{f_0(x)-b_0\}\cos\beta_m x dx = b_m\frac{c}{2}\left\{1+\frac{\sin 2\beta_m c}{2\beta_m c}\right\}$$
$$+\sum_{k\neq m} b_k \frac{1}{\beta_k^2-\beta_m^2}\left\{\frac{\psi(-\beta_k^2)}{\chi(-\beta_k^2)}-\frac{\psi(-\beta_m^2)}{\chi(-\beta_m^2)}\right\}\cos\beta_k c\cos\beta_m c.$$

The most general case in which this equation suffices to determine the coefficient b_m is that in which $\dfrac{\psi(-\beta^2)}{\chi(-\beta^2)}=p+q\beta^2$, where p and q are constants.

The equation then becomes

$$\int_0^c \{f_0(x)-b_0\}\cos\beta_m x dx = b_m\frac{c}{2}\left\{1+\frac{\sin 2\beta_m c}{2\beta_m c}\right\}$$
$$+\sum_{k\neq m} b_k q\cos\beta_k c\cos\beta_m c$$
$$=b_m\frac{c}{2}\left\{1+\frac{\sin 2\beta_m c}{2\beta_m c}\right\}+q\cos\beta_m c\Sigma b_k\cos\beta_k c-qb_m\cos^2\beta_m c$$
$$=b_m\frac{c}{2}\left\{1+\frac{\sin 2\beta_m c}{2\beta_m c}\right\}-qb_m\cos^2\beta_m c+q\{f_0(c)-b_0\}\cos\beta_m c;$$

therefore

$$b_m=\frac{\int_0^c\{f_0(x)-b_0\}\cos\beta_m x dx-q\{f_0(c)-b_0\}\cos\beta_m c}{\frac{c}{2}\left\{1+\frac{\sin 2\beta_m c}{2\beta_m c}\right\}-q\cos^2\beta_m c}.$$

Since (3) now becomes

$$(p+q\beta^2)\cos\beta c-\beta\sin\beta c=0,$$

we have $$\left(\frac{p}{\beta^2}+q\right)\cos^2\beta c-\frac{1}{\beta}\sin\beta c\cos\beta c=0,$$

or $$\left(\frac{p}{\beta^2}+q\right)\cos^2\beta c-\frac{\sin 2\beta c}{2\beta}=0.$$

Hence

$$\frac{c}{2}+\frac{\sin 2\beta_m c}{4\beta_m}-q\cos^2\beta_m c=\frac{c}{2}+\frac{1}{2}\left(\frac{p}{\beta_m^2}-q\right)\cos^2\beta_m c,$$

and

$$b_m=2\frac{\int_0^c\{f_0(x)-b_0\}\cos\beta_m x\,dx-q\{f_0(c)-b_0\}\cos\beta_m c}{c+\left(\frac{p}{\beta_m^2}-q\right)\cos^2\beta_m c}\quad\ldots(5).$$

Dealing now with

$$f_1(x)=a_1\sin\alpha_1 x+a_2\sin\alpha_2 x+a_3\sin\alpha_3 x+\&c.,$$

we have, on multiplying by $\sin\alpha_m x$ and integrating between the limits 0 and c,

$$\int_0^c f_1(x)\sin\alpha_m x\,dx=a_m\int_0^c\sin^2\alpha_m x\,dx+\sum_{k\neq m}a_k\int_0^c\sin\alpha_k x\sin\alpha_m x\,dx,$$

where the summation in the last member extends over all terms for which k is not equal to m,

$$=a_m\frac{c}{2}\left\{1-\frac{\sin 2\alpha_m c}{2\alpha_m c}\right\}$$

$$+\sum_{k\neq m}a_k\frac{1}{\alpha_k^2-\alpha_m^2}\{-\alpha_k\cos\alpha_k c\sin\alpha_m c+\alpha_m\sin\alpha_k c\cos\alpha_m c\}$$

$$=a_m\frac{c}{2}\left\{1-\frac{\sin 2\alpha_m c}{2\alpha_m c}\right\}$$

$$+\sum_{k\neq m}a_k\frac{1}{\alpha_k^2-\alpha_m^2}\left\{\frac{\psi(-\alpha_k^2)}{\chi(-\alpha_k^2)}-\frac{\psi(-\alpha_m^2)}{\chi(-\alpha_m^2)}\right\}\sin\alpha_k c\sin\alpha_m c,$$

which reduces, since $\dfrac{\psi(-\alpha^2)}{\chi(-\alpha^2)}=p+q\alpha^2$, to

$$\int_0^c f_1(x)\sin\alpha_m x\,dx=a_m\frac{c}{2}\left\{1-\frac{\sin 2\alpha_m c}{2\alpha_m c}\right\}+\sum_{k\neq m}a_k q\sin\alpha_k c\sin\alpha_m c$$

$$= a_m \frac{c}{2}\left\{1 - \frac{\sin 2\alpha_m c}{2\alpha_m c}\right\} + q \sin\alpha_m c \Sigma a_k \sin\alpha_k c - q a_m \sin^2\alpha_m c$$

$$= a_m \left\{\frac{c}{2}\left(1 - \frac{\sin 2\alpha_m c}{2\alpha_m c}\right) - q \sin^2\alpha_m c\right\} + q f_1(c) \sin\alpha\ c;$$

therefore $$a_m = \frac{\int_0^c f_1(x) \sin\alpha_m x dx - q f_1(c) \sin\alpha_m c}{\frac{c}{2}\left\{1 - \frac{\sin 2\alpha_m c}{2\alpha_m c}\right\} - q \sin^2\alpha_m c}.^*$$

Since (3) now becomes

$$(p + q\alpha^2) \sin\alpha c + \alpha \cos\alpha c = 0,$$

we have $$\left(\frac{p}{\alpha^2} + q\right) \sin^2\alpha c + \frac{1}{\alpha} \sin\alpha c \cos\alpha c = 0,$$

or $$\left(\frac{p}{\alpha^2} + q\right) \sin^2\alpha c + \frac{\sin 2\alpha c}{2\alpha} = 0.$$

Hence

$$\frac{c}{2} - \frac{\sin 2\alpha c}{4\alpha} - q \sin^2\alpha c = \frac{c}{2} + \frac{1}{2}\left(\frac{p}{\alpha^2} - q\right) \sin^2\alpha c,$$

and $$a_m = 2 \frac{\int_0^c f_1(x) \sin\alpha_m x dx - q f_1(c) \sin\alpha_m c}{c + \left(\frac{p}{\alpha_m^2} - q\right) \sin^2\alpha_m c} \quad \ldots\ldots\ldots (6).$$

Hence, it is proved that if

$$y = f(x) = b_0 + b_1 \cos\beta_1 x + b_2 \cos\beta_2 x + \ldots \quad \{= f_0(x)\}$$
$$+ a_1 \sin\alpha_1 x + a_2 \sin\alpha_2 x + \ldots \quad \{= f_1(x)\} \ldots (1),$$

series which are convergent between the limits $x = 0$ and $x = c$, where the α's and β's are such that at $x = c$ they satisfy the condition

$$\phi(D)\, y = C,$$

where $$D \equiv \frac{dx}{d}, \ \phi(D) \equiv \psi(D^2) + D\chi(D^2),$$

and $$\frac{\psi(D^2)}{\chi(D^2)} = p - qD^2,$$

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* Stephenson omits the term $q \sin^2\alpha_m$ from the denominator of his result, p. 77.

i.e. the condition $(p - qD^2 + D)\,\chi(D^2)\,y = C$, when $x = c$;

then the coefficients a and b are determined by the equations

$$a_m = 2\,\frac{\int_0^c f_1(x)\sin\alpha_m x\,dx - qf_1(c)\sin\alpha_m c}{c + \left(\frac{p}{\alpha_m^2} - q\right)\sin^2\alpha_m c}$$

$$b_n = \frac{C}{A_0 + A_1 c + A_2\frac{c^2}{2} + \dots}$$

$$b_m = 2\,\frac{\int_0^c \{f_0(x) - b_0\}\cos\beta_m x\,dx - q\,\{f_0(c) - b_0\}\cos\beta_m c.}{c + \left(\frac{p}{\beta_m^2} - q\right)\cos^2\beta_m c}$$

These expressions hold so long as the series (1) and its differential coefficients, up to the highest which enters into $\phi(D)$, are convergent within the range $x = 0$ to $x = c$.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy*, Trinity College, Cambridge.

XVIII.

On some discontinuous integrals.

§ 1. The definite integral

$$(1) \qquad \int_0^\infty \frac{\sin(ax - b\sin x)}{x}\,dx,$$

in which a and b are real, possesses interesting properties. In the first place its convergence does not follow immediately from any of the tests usually given in the books. It may, however, be deduced from the convergence theorem given in Note II.

For

$$\int_0^{2m\pi} \sin(ax - b\sin x)\,dx = \sum_{k=0}^{m-1} \int_0^{2\pi} \sin(ax + 2ak\pi - b\sin x)\,dx$$

$$= \sum_{k=0}^{m-1} \left\{ \sin 2ak\pi \int_0^{2\pi} \cos(ax - b\sin x)\,dx \right.$$

$$\left. + \cos 2ak\pi \int_0^{2\pi} \sin(ax - b\sin x)\,dx \right\},$$

and hence, provided a is not an integer,

$$\left| \int_0^{2m\pi} \sin(ax - b\sin x)\,dx \right| < K,$$

where K is a constant. Hence the limits of indetermination of

$$(2) \qquad \int_0^x \sin(ax - b\sin x)\,dx,$$

for $x = \infty$, are finite; and therefore, by the theorem referred to above, the integral (1) is convergent.

If, however, a is an integer,

$$\int_0^{2m\pi} \sin(ax - b\sin x)\,dx = \int_0^{2m\pi} \sin\{a(2m\pi - x) + b\sin x\}\,dx$$

(putting $2m\pi - x$ for x)

$$= -\int_0^{2m\pi} \sin(ax - b\sin x)\,dx,$$

and therefore $= 0$. Hence again the limits of indetermination of (2) are finite, and (1) is convergent.

§ 2. In the second place the integral is discontinuous for for all integral values of a. For, since

$$\sin(b\sin x) = \sum_{-\infty}^{\infty} J_n(b)\sin nx,$$

$$\cos(b\sin x) = \sum_{-\infty}^{\infty} J_n(b)\cos nx,$$

it follows that

$$\sin(ax - b\sin x) = \sum_{-\infty}^{\infty} J_n(b)\sin(a - n)x.$$

The series on the right is uniformly convergent in any finite interval $(0, X)$ of values of x. Hence

$$\int_0^X \frac{\sin(ax - b\sin x)}{x}\,dx = \sum_{-\infty}^{\infty} J_n(b) \int_0^X \frac{\sin(a-n)\,x}{x}\,dx.$$

In this equation we may replace X by ∞ if we can show that

$$\lim_{X=\infty} \sum_{-\infty}^{\infty} J_n(b) \int_X^{\infty} \frac{\sin(a-n)\,x}{x}\,dx = 0.$$

Suppose that a is not an integer.* Let $a - n = a_n$, so that a_n is always greater numerically than some constant δ. Suppose $a_n > 0$, and let $\dfrac{m\pi}{a_n}$ be the multiple of $\dfrac{\pi}{a_n}$ equal to or immediately greater than X. Then

$$\int_X^{\infty} \frac{\sin a_n x}{x}\,dx = \int_{a_n X}^{\infty} \frac{\sin u}{u}\,du = \left(\int_{a_n X}^{m\pi} + \sum_{k=m}^{\infty} \int_{k\pi}^{(k+1)\pi}\right) \frac{\sin u}{u}\,du.$$

Now
$$\left|\int_{a_n X}^{m\pi} \frac{\sin u}{u}\,du\right| < \frac{K}{(m-1)\pi},$$

where K is a constant. Also

$$\sum_{k=m}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{\sin u}{u}\,du = \sum_{k=m}^{\infty} (-)^k \int_0^{\pi} \frac{\sin u}{u + k\pi}\,du.$$

Now the series

$$\sum_{k=m}^{\infty} \frac{(-)^k}{u + k\pi}$$

is uniformly convergent for all values of u in $(0, \pi)$, and is in absolute value $< \dfrac{1}{m\pi}$. Hence

$$\left|\sum_{k=m}^{\infty} (-)^k \int_0^{\pi} \frac{\sin u}{u + k\pi}\,du\right| = \left|\int_0^{\pi} \sin u \sum_{k=m}^{\infty} \frac{(-)^k}{u + k\pi}\,du\right| < \frac{2}{m\pi},$$

and so
$$\left|\int_X^{\infty} \frac{\sin a_n x}{x}\,dx\right| < \frac{K'}{m\pi},$$

where K' is another constant.

Also since
$$(m-1)\,\pi < a_n X \leq m\pi,$$

and $a_n > \delta$, it follows that $m > CX$,

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* No particular difficulty arises when a is an integer.

and so
$$\left|\int_X^\infty \frac{\sin a_n x}{x}\,dx\right| < \frac{H}{X},$$
where C, H are further constants. The same may be proved when a_n is negative.

Also since
$$\lim_{n=\infty} \frac{J_{n+1}}{J_n} = \lim_{n=\infty} \frac{J_{-n-1}}{J_{-n}} = 0,$$
the series
$$\Sigma J_n(b) \int_X^\infty \frac{\sin(a-n)x}{x}\,dx$$
is absolutely convergent and $<\dfrac{L}{X}$, where L is another constant. Its limit for $X=\infty$ is therefore 0. Hence
$$\int_0^\infty \frac{\sin(ax-b\sin x)}{x}\,dx = \sum_{-\infty}^{\infty} J_n(b)\int_0^\infty \frac{\sin(a-n)x}{x}\,dx = \tfrac{1}{2}\pi\sum_{-\infty}^{\infty}\epsilon_n J_n(b),$$
where $\epsilon_n = 1$, 0, or -1, according as $a-n$ is positive, zero, or negative.

It follows that the integral (1) is discontinuous for $a=n$, the magnitude of its discontinuity being πJ_n, and its value for $a=n$ the mean of its values for $a=n-0$ and $a=n+0$. It is evident that the same reasoning applies to any integral of the form
$$\int_0^\infty \frac{\phi(a,\,x)}{x}\,dx,$$
where
$$\phi(a,\,x) = \sum_{-\infty}^{\infty} c_n \sin(a-n)x,$$
and Σc_n is absolutely convergent.

§ 3. The integral (1) may also be evaluated in another way, which leads to a different form of the result.

Suppose $0<a<1$: then
$$\int_0^\infty \sin(ax-b\sin x)\frac{dx}{x} = \frac{1}{2}\int_{-\infty}^{\infty} = \frac{1}{2}\sum_{-\infty}^{\infty}\int_{2n\pi}^{2(n+1)\pi}$$
$$= \frac{1}{2}\sum_{\infty}^{\infty}\int_0^{2\pi} \frac{\sin(ax-b\sin x+2na\pi)}{x+2n\pi}\,dx,$$
and it is not difficult to prove that the sign of summation may be transferred under the integral sign.

Now,* if $0<v<1$,
$$\sum_{-\infty}^{\infty}\frac{e^{2nv\pi i}}{w-n} = \frac{2\pi i e^{2wv\pi i}}{e^{2w\pi i}-1}.$$

* Kronecker, *Vorlesungen ueber Integrale*, p. 105.

Putting $v=a$ and $w=-\dfrac{x}{2\pi}$, we find that

$$\sum_{-\infty}^{\infty}\frac{e^{2na\pi i}}{x+2n\pi}=-i\frac{e^{-axi}}{e^{-xi}-1}=\frac{e^{(\frac{1}{2}-a)xi}}{2\sin\frac{1}{2}x},$$

and hence that

$$\sum_{-\infty}^{\infty}\frac{\sin(ax-b\sin x+2na\pi)}{x+2n\pi}=\frac{\sin(\frac{1}{2}x-b\sin x)}{2\sin\frac{1}{2}x};$$

and so our integral is equal to

$$u=\tfrac{1}{4}\int_0^{2\pi}\frac{\sin(\frac{1}{2}x-b\sin x)}{\sin\frac{1}{2}x}\,dx.$$

Now

$$\frac{du}{db}=-\tfrac{1}{2}\int_0^{2\pi}\cos(\tfrac{1}{2}x-b\sin x)\cos\tfrac{1}{2}x\,dx=-\tfrac{1}{4}\int_0^{2\pi}\cos(x-b\sin x)\,dx$$

$$-\tfrac{1}{4}\int_0^{2\pi}\cos(b\sin x)\,dx=-\tfrac{1}{2}\pi(J_0+J_1).$$

Hence $$u=\tfrac{1}{2}\pi\int_b(J_0+J_1)\,db.$$

Now it is easy to see that u is continuous for $b=0$ and equal to $\frac{1}{2}\pi$. Hence

$$u=\tfrac{1}{2}\pi\left\{1-\int_0^b(J_0+J_1)\,db\right\},$$

which may also be written in either of the forms

$$u=\tfrac{1}{2}\pi\left\{\int_b^\infty J_0(b)\,db-\int_0^b J_1(b)\,db\right\},$$

$$u=\tfrac{1}{2}\pi\left\{\int_b^\infty J_1(b)\,db-\int_0^b J_0(b)\,db\right\},$$

as $$\int_0^\infty J_0(b)\,db=\int_0^\infty J_1(b)\,db=1.$$

§ 4. We can verify the identity of our two formulæ by means of the relation

$$J_n'=\tfrac{1}{2}(J_{n-1}-J_{n+1}).$$

For $\quad J_0' + J_{-1}' + J_{-2}' + J_{-3}' + \ldots - J_1' - J_2' - J_3' - \ldots$

$= \frac{1}{2} \{J_{-1} - J_1 + J_{-2} - J_0 + J_{-3} - J_{-1} + J_{-4} - J_{-2} + \ldots$

$$- J_0 + J_2 - J_1 + J_3 - J_2 + J_4 - \ldots\} = - J_1 - J_0,$$

so that $\quad J_0 + J_{-1} + J_{-2} \ldots - J_1 - \ldots = \int_b (J_0 + J_1)\, db,$

and as the left-hand side has the value 1 for $b = 0$,

$$J_0 + J_{-1} + J_{-2} \cdots - J_1 - \ldots = 1 - \int_0^b (J_0 + J_1)\, db.$$

§ 5. Returning now to the investigation of § 3, suppose $0 < p < a < p + 1$, where p is an integer, positive or negative: then

$$\sum_{-\infty}^{\infty} \frac{e^{2na\pi i}}{x + 2n\pi} = \sum_{-\infty}^{\infty} \frac{e^{2n(a-p)\pi i}}{n + 2n\pi} = \frac{e^{(p + \frac{1}{2} - a)xi}}{2 \sin \frac{1}{2} x},$$

so that our integral is equal to

$$u = \frac{1}{4} \int_0^{2\pi} \frac{\sin \{(p + \frac{1}{2}) x - b \sin x\}}{\sin \frac{1}{2} x}\, dx.$$

Now

$$\frac{du}{db} = - \frac{1}{2} \int_0^{2\pi} \cos \{(p + \tfrac{1}{2}) x - b \sin x\} \cos \tfrac{1}{2} x dx$$

$$= - \frac{1}{4} \int_0^{2\pi} \cos (px - b \sin x)\, dx$$

$$- \frac{1}{4} \int_0^{2\pi} \cos \{(p + 1) x - b \sin x\}\, dx = - \tfrac{1}{2}\pi (J_p + J_{p+1}).$$

Thus $\quad u = \frac{1}{2}\pi \int_b (J_p + J_{p+1})\, db.$

But for $b = 0$,

$$u = \frac{1}{4} \int_0^{2\pi} \frac{\sin (p + \frac{1}{2}) x}{\sin \frac{1}{2} x}\, dx = \frac{1}{2} \int_0^{\pi} \frac{\sin (2p + 1) u}{\sin u}\, du = \tfrac{1}{2}\pi,$$

and so $\quad u = \frac{1}{2}\pi \left\{1 - \int_0^b (J_p + J_{p+1})\, db\right\}.$

This again is easily identified with the former value for u. If $a = p$ this method fails, but it is easy to see that

$$u = \tfrac{1}{2}\pi \left\{1 - \frac{1}{2} \int_0^b (J_{p-1} + 2J_p + J_{p+1})\, db.\right\}$$

If, however, p is negative,

$$\int_0^\pi \frac{\sin(2p+1)u}{\sin u}\,du = -\pi,$$

so that $u = -\tfrac{1}{2}\pi$ for $b=0$, and

$$u = \tfrac{1}{2}\pi\left\{-1 - \int_0^b (J_p + J_{p+1})\,db\right\},$$

while, for $a = p$,

$$u = \tfrac{1}{2}\pi\left\{-1 - \tfrac{1}{2}\int_0^b (J_{p-1} + 2J_p + J_{p+1})\,db.\right.$$

These results again accord with that given by the series for u, since the coefficient of J_0 is now negative. Finally, when $a=0$,

$$u = -\tfrac{1}{2}\pi\int_0^b J_0(b)\,db.$$

The value of u is thus determined for all real values of a and b.

§ 6. There is one point on which it may be worth while to make a few remarks. Let us take the case in which $a=0$ for simplicity. In this case

$$\int_0^\infty \frac{\sin(b\sin x)}{x}\,dx = \tfrac{1}{2}\pi\int_0^b J_0(b)\,db,$$

or

$$= -J_{-1} - J_{-2} + \ldots + J_1 + J_2 \ldots$$

$$= 2(J_1 + J_3 + \ldots).$$

The limit of the integral for $b=\infty$ is

$$\tfrac{1}{2}\pi\int_0^\infty J_0(b)\,db = \tfrac{1}{2}\pi,$$

although each of the terms in the series $J_1 + J_3 + \ldots$ vanishes for $b=\infty$. Or, in other words, the equation

$$2J_1' + 2J_3' + \ldots = J_0$$

cannot be integrated term by term from $b=0$ to $b=\infty$. The same is true of whole classes of series of Bessel functions; such as

$$1 = J_0 + 2J_2 + 2J_4 + \ldots,$$

$$0 = J_0' + 2J_2' + 2J_4' + \ldots.$$

§ 7. It is easy to verify that the limit of

$$u = \tfrac{1}{4}\int_0^{2\pi} \frac{\sin(\frac{1}{2}x - b\sin x)}{\sin\frac{1}{2}x}\,dx$$

for $b=\infty$ is in fact $-\frac{1}{2}\pi$, and not 0. For

$$u = \tfrac{1}{4}\pi\int_0^{2\pi}\cos(b\sin x)\,dx - \tfrac{1}{4}\pi\int_0^{2\pi}\cot\tfrac{1}{2}x\sin(b\sin x)\,dx,$$

and the limit of the first term for $b=\infty$ is 0. As to the second we divide it into three integrals between the limits

$$(0,\ \epsilon)\ (\epsilon,\ 2\pi-\epsilon)\ (2\pi-\epsilon,\ 2\pi).$$

It can be shown that the limit of the second integral is 0. To prove that the limit of the first is π we argue (roughly) as follows. We replace

$$\int_0^{\epsilon}\cot\tfrac{1}{2}x\sin(b\sin x)\,dx$$

by
$$2\int_0^{\epsilon}\frac{\sin bx}{x}\,dx = 2\int_0^{b\epsilon}\frac{\sin u}{u}\,du,$$

the limit of which is π. Similarly for the third integral, and the result follows.

§ 8. The preceding results may be employed to construct integrals which are discontinuous for all rational values of a parameter. We saw that

$$\int_0^{\infty}\sin(ax - b\sin x)\frac{dx}{x}\quad (a>0)$$

is discontinuous for all integral values of a, the discontinuity at $a=n$ being πJ_n. Now consider the series

$$\sum_0^{\infty}\frac{\sin(nax - b\sin x)}{n!},$$

the value of which is easily found to be

$$e^{\cos ax}\sin(\sin ax - b\sin x).$$

From this we infer that the integral

$$\int_0^{\infty} e^{\cos ax}\frac{\sin(\sin ax - b\sin x)}{x}\,dx$$

is discontinuous for all *rational* values of a. For suppose $a=\dfrac{p}{q}$.

The integral is the sum of a series of terms of which the $q\lambda^{\text{th}}$ ($\lambda = 0, 1, 2, \ldots$) is discontinuous for $a = \frac{p}{q}$, the magnitude of the discontinuity being

$$\frac{\pi J_{\lambda p}}{(\lambda q)\,!},$$

and the aggregate discontinuity

$$\pi \sum_1^\infty \frac{J_{\lambda p}(b)}{(\lambda q)\,!}.$$

Owing to the length which this note has already assumed I shall not set out the formal proof, the general lines of which have been indicated above. Instead of

$$\Sigma \frac{\sin(nax - b\sin x)}{n\,!},$$

we might have used any one of a whole class of series

$$\Sigma c_n \sin(nax - b\sin x).$$

These integrals have then a property analogous to that of the integrals considered in Note XIV. They are less simple, but have the advantage of being ordinary integrals and not principal values.

HIGH PELLIAN FACTORISATIONS.

[Shewing factorisation of $N = y^2 + 1$, up to 78 figures.]

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's College, London.

[The author is indebted to Mr. H. J. Woodall, A.R.C.Sc. for help in reading the proof-sheets, and for some useful suggestions.]

1. *Pellian Factorisation.* By this term is here meant such factorisations of numbers (N), of form $N = y^2 + 1$, as are easily derivable from (known) solutions of the Pellian Equation

$$y^2 - D.x^2 = -1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Every solution of this equation gives *at sight* a factorisation of the number

$$N = y^2 + 1 = D.x^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The interest of this application lies chiefly in the following points:—

1°. The factorisable numbers (N) all contain a *square factor* (x^2).

2°. Even when D is small (say < 1500), the *minimum* solutions (say x_1, y_1) of (1) are frequently *large* (and sometimes *very large*) numbers, so that immense factorisable numbers (N) are often obtained, containing large square factors.

3°. From any (known) solutions (x_1, y_1) new solutions in higher numbers (x_r, y_r) are readily derived, yielding still higher factorisable numbers (N_r).

4°. The existing Tables of the Pellian Equation solutions yield *at sight* the factorisation of a good many very high numbers (N).

The numbers N, factorisable in this manner, will be here styled *Pellian Numbers.*

1*a*. *Notation.* In this Paper ω denotes an *odd* number.

2. *Successive solutions.* Let (y_1, x_1), (η_1, ξ_1) be *minimum* solutions

$$y_1, x_1 \text{ of } y^2 - D.x^2 = -1 \text{ ....(1)},$$

$$\eta_1, \xi_1 \text{ of } \eta^2 - D.\xi^2 = +1 \text{ ....(3)},$$

Then $$\eta_1 = y_1^2 + D.x_1^2, \quad \xi_1 = 2x_1y_1 \text{............(4)},$$

The *next* solution (say y_2, x_2), of Eq. (1) is given by

$$y_2 = \eta_1 y_1 + D.\xi_1 x_1, \quad x_2 = \xi_1 y_1 + \eta_1 x_1 \text{.........(5)},$$

and the whole series of solutions, say (y_3, x_3), ... (y_r, x_r) of Eq. (1) may now be derived by *successive* applications of either set of the general formulæ (6) or (7),

$$y_r = \eta_1 y_{r-1} + D.\xi_1 x_{r-1}, \quad x_r = \xi_1 y_{r-1} + \eta_1 x_{r-1} \text{......(6)},$$

$$y_r = 2\eta_1 y_{r-1} - y_{r-2}, \qquad x_r = 2\eta_1 x_{r-1} - x_{r-2} \text{.......(7)}.$$

In terms of the original y_1, x_1, the next two solutions (y_2, x_2), (y_3, x_3) are

$$y_2 = y_1(y_1^2 + 3D.x_1^2), \qquad x_2 = x_1(3y_1^2 + D.x_1^2) \text{.........(8)},$$

$$y_3 = y_1(y_1^4 + 10D.y_1^2x_1^2 + 5D^2x_1^4),\ x_3 = x_1(5y_1^4 + 10Dy_1^2x_1^2 + D^2x_1^4)\text{.(9)},$$

These give the factorisation of the successive Pellian numbers N_1, N_2, ..., N_r, of type

$$N_r = y_r^2 + 1 = D.x_r^2 \text{ ....(10)}.$$

This process may be pushed *to any extent.* The solutions (y_r, x_r) increase rapidly in magnitude; in fact formula (7) shows that $y_r > (2\eta_1)^{r-1}$, so that the factorisable numbers N_r rapidly become enormous (even when D, y_1, x_1 are small).

3. *Complete Factorisation.* As, by the above process large numbers of form $N = (y^2 + 1)$ can be readily evolved of *any desired magnitude*, along with their *algebraic* resolution into *three* factors (D, x_r, x_r), the chief interest lies in producing large numbers (N) which shall be *completely factorisable*‡ (*i.e.* into prime factors). This depends on the factorisability of both D, x_r; but chiefly on that of x_r, because N contains x_r^2, and x_r itself is usually $> D$.

4. *Factorisation-limits.* The ordinary limit of complete factorisability of D and x_r is—except when D, or x_r are of special forms—simply that of the large Factor-Tables, which extend (at present*) to 9 millions, with a short extension† to 9,001,020; so that the ordinary limit of complete factorisability of Pellian numbers is

$$N < \text{about } (9 \text{ million})^3, \text{ or } \not> 21 \text{ figures.}$$

But, by choosing D, x_r of special forms (Art. 4, 5, 6), and especially by choosing x_r of a form which is itself resolvable *algebraically* (Art. 7), this limit may be greatly exceeded, as N will thus be resolved into at least *five* (algebraic) factors. Further extension of the limits may be obtained by arranging that one, or both, of D, x_r shall be of form $(y_r^2 + 1)$ for which special Tables exist (Art 4*b*–*f*), or of form $(y^{2r} + 1)$ whose complete factorisation is known for small values of y (Art. 8–11).

[In the Examples of completely factorised High Pellian Numbers (N) which follow this, the *number of figures* contained in N is entered in square brackets at the end of each Example, to show the *power*‡ of the process].

4*a*. *Form* $N = y^2 + 1$. The factorisable numbers N in this Paper, being all of form $N = y^2 + 1$, their factorisation is much aided when they have factors (N) which are also of form $N = y^2 + 1$.

4*b*. *By the Factor-Tables.* Since $N = (y^2 + 1)$ has the factors 2, 5, 10 in following cases

$$\begin{array}{llll} N = y^2 + 1 & \equiv 0 \pmod 2; & \equiv 0 \pmod 5; & \equiv 0 \pmod{10} \\ \text{when} & y \text{ is } odd\;; & y = 10\eta \pm 2, 3\,; & y = 10\eta \pm 3, \end{array}$$

* The *Bulletin of the American Mathematical Society* of July, 1903, contains (p. 539) a Notice of a *new* set of Factor-Tables under preparation by Dr. D. N. Lehmer of the University of California, which is projected to extend to 10 million.

† A Table of the Least Factors of the numbers N ($\neq 2n, 3n, 5n$) from 9.10^6 to $9.10^6 + 1020$ is given in a Paper on *Determination of Successive High Primes*, by the present author and Mr. H. J. Woodall (jointly) in *Messenger of Mathematics*, Vol XXXIV., 1901, Tab. V., p. 80.

‡ Note that *all* the factorisations in this Paper are *complete* (i.e. into prime factors).

it follows that the large Factor-Tables give *complete* factorisation of $N = (y^2 + 1)$ up to the limits (y_m) shewn

$y = 10\eta,\ 10\eta \pm 4$;	$10\eta \pm 1, 5$;	$10\eta \pm 2$;	$10\eta \pm 3$
Limit, $y_m =$ 3000 ;	4241 ;	6708 ;	9483.

4*c*. *Factor-Table of* $(y^2 + 1)$. The author has prepared a Table* giving the *complete* factorisation of $N = y^2 + 1$; this is now *continuous* up to $y = 15000$.

4*d*. *Congruence Tables of* $y^2 + 1 \equiv 0$. In the course of making the Factor-Table of $(y^2 + 1)$, the author has prepared a Table* giving the *two* roots (y, y') $[< p \text{ or } p^\kappa]$ of the congruence

$$y^2 + 1 \equiv 0 \ (\text{mod}\ p \text{ or } p^\kappa),$$

This is now complete for all primes (p) and powers of primes $(p^\kappa) \not> 33000$†, and also for very many other values of p and $p^\kappa \not> 100000$. By casting out the divisors (p and p^κ) shown by this Table, it is possible to extend the factorisation of $N = y^2 + 1$ *continuously* up to the limits (y_m),

$y = 10\eta,\ 10\eta \pm 4$;	$10\eta \pm 1, 5$;	$10\eta \pm 2$;	$10\eta \pm 3$
Limit, $y_m =$ 33010 ;	46685 ;	73812 ;	104393,

and, in many cases far beyond.

4*e*. *Factor Table of* $(y^4 + 1)$. The author has also prepared a Table‡ giving the *complete* factorisation of $N = y^4 + 1$; this is now *continuous* up to $y = 202$ (*y even*), and $y = 249$ (*y odd*), and also extends (with many breaks) up to $y = 1000$.

4*f*. *Congruence-Table of* $y^4 + 1 \equiv 0$. In the course of making the above Factor-Table‡ of $N = y^4 + 1$, the author has

* These two Tables, the Factor-Table of $N = y^2 + 1$, and the Congruence-Table of $y^2 + 1 \equiv 0$, are at present only in MS. Their construction and use have been explained in the Author's Paper on *High Primes* $(4\varpi + 1)$, $(6\varpi + 1)$ *and Factorisations* in the *Quarterly Journal*. Vol. XXXV., 1903, pp. 10—21. They enable the detection of *all* High Primes $p = 4\varpi + 1$ of 7, 8, 9, 10 figures (up to 1089 million), when occurring as factors of $N = y^2 + 1$. They were prepared by the Author's Assistants (Miss B. E. Haselden, and Miss B. B. Haselden) under his constant superintendence.

† Extended to 50000 since this was in type.

‡ These two Tables, the Factor-Table of $N = y^4 + 1$, and the Congruence-Table of $y^4 + 1 \equiv 0$, are at present only in MS. They enable the detection of *all* High Primes $p = 8\varpi + 1$ of 7, 8, 9, or 10 figures ($\not> 1050$ million), when occurring as factors of $N = y^4 + 1$. They were computed partly by the Author himself, partly by his Assistants (Miss A. B. Cole, and Miss E. Cooper) under his constant superintendence.

also prepared a Table giving the *four* roots (y, y', y'', y''') $[< p$ or $p^\kappa]$ of the congruence

$$y^4 + 1 \equiv 0 \,(\text{mod}\, p \text{ and } p^\kappa).$$

This is now complete for all primes (p) and powers of primes $(p^\kappa) \not> 32441$, and also for very many other values of p and $p^\kappa \not> 100000$. By casting out the divisors (p and p^κ) shewn by this Table, it is possible to extend the factorisation of $N = y^4 + 1$ in many cases far beyond the limit $y = 1000$ of the Factorisation Table already made (sometimes even beyond $y = 10^5$).

5. *Simplest* $D = (y^2 + 1)$. The simplest form of D is $D = y^2 + 1$, as it has the (obvious) minimum solution, $y^2 - D \,.\, 1^2 = -1$, whence, by (1), (3),

$$y_1 = y,\ x_1 = 1\ ;\quad \eta_1 = 2y^2 + 1,\ \ \xi_1 = 2y \ldots\ldots\ldots (11).$$

Hence follow the first two successive derived solutions; by (8), (9),

$$y_2 = y(4y^2 + 3), \qquad\qquad x_2 = (4y^2 + 1) \ldots\ldots\ldots\ldots (12),$$
$$y_3 = y\,(16y^4 + 20y^2 + 5),\ x_3 = (16y^4 + 12y^2 + 1) \ldots (13).$$

6. *Simplest Pellian Form.* The first of the above "derived solutions" gives

$$\mathrm{N} = (4y^3 + 3y)^2 + 1 = D \,.\, x_2^{\,2} = (y^2 + 1)\,(4y^2 + 1)^2 \ldots (14),$$

a form of considerable importance, as it leads readily to *very high* factorisable numbers (N), in consequence of the two factors (D, x_r) being of the *same* form $(Y^2 + 1)$, shewn to be specially suitable (Art. 4*a*–*d*).

[It is here supposed that $y \neq \square$, this case being deferred to Art. 7].

6*a*. *By the Factor-Tables.* These give complete factorisation of *all* these N up to the limits, y_m (Art. 4*b*),

$$y = 10\eta,\ 10\eta \pm 2,\ 3,\ 5\ ;\quad 10\eta \pm 4\ ;\quad 10\eta \pm 1$$
$$y_m = \qquad 1500 \qquad\quad ;\quad 2996 \quad ;\quad 3351.$$

Examples.

$(4.1500^3 + 3.1500)^2 + 1 = (1500^2 + 1)\,(3000^2 + 1)^2 = (13.17.10181)\,(61.147541)^2$;
[21 figs.]

$(4.2996^3 + 3.2996)^2 + 1 = (2996^2 + 1)\,(5992^2 + 1)^2 = (17.528001)\,(5.97.181.409)^2$;
[23 figs.]

$(4.3351^3 + 3.3351)^2 + 1 = (3351^2 + 1)\,(6702^2 + 1)^2 = (2.233.24097)\,(5.17.528433)^2$;
[23 figs.]

6*b*. *By the Factorisation Tables of* (y^2+1), [Art. 4*c*]. This gives complete factorisation (at sight) of *all* these **N** up to the limit $y_m = 7500$ (Art. 4*c*).

$$(4.7500^3+3.7500)^2+1=(7500^2+1)(15000^2+1)^2=(197.28533)(12433.18097)^2;$$
[25 figs.]

Interesting cases arise when the factors (D, x_2) are *both* High Primes (> 9 million); the highest of these in this Table is

$$(4.7190^3+3.7190)^2+1=(7190^2+1)(14380^2+1)^2=51696101.(206784401)^2;$$
[25 figs.]

6*c*. *By the Congruence-Table of* $y^2+1\equiv 0$, [Art. 4*d*]. By casting out the divisors shewn in this Table, *complete* factorisation of *all* these numbers (N) can be obtained up to the limits y_m, (Art. 4*d*).

$y =$	$10\eta,\ 10\eta\pm 2,\ 3,\ 5$;	$10\eta\pm 4$;	$10\eta\pm 1$;
$y_m =$	16505 ;	33006 ;	36901 ;

Examples.

$$(4.16505^3+3.16505)^2+1=(16505^2+1)(33010^2+1)^2$$
$$=(2.13.1913.5477)(17.37.1732369)^2; \quad [27 \text{ figs.}]$$

$$(4.33006^3+3.33006)^2+1=(33006^2+1)(66012^2+1)^2$$
$$=(1089396037)(5.673.1294973)^2; \quad [29 \text{ figs.}]$$

$$(4.36901^3+3.36901)^2+1=(36901^2+1)(73802^2+1)^2$$
$$=(2.89.277.27617)(5.23977.45433)^2; \quad [29 \text{ figs.}]$$

7. *Bin-Aurifeuillian Extension.* By taking $y_1 = y^2$, in Art. 5, 6,

$$D=y^4+1,\ y_1=y^2,\ x_1=1;\quad \eta_1=2y^4+1,\ \xi_1=2y^2 \ldots(15);$$
$$y_2=y^2(4y^4+3),\ x_2=4y^4+1 \ldots\ldots\ldots\ldots\ldots(16).$$

Here x_2 is a *Bin-Aurifeuillian** resolvable *algebraically* into the two factors

$$x_2=L.M;\quad L=(y-1)^2+y^2,\ M=(y+1)^2+y^2 \ldots(17),$$

and **N** becomes

$$\mathbf{N}=(4y^6+3y^2)^2+1=D.x_2^{\,2}=\dagger D.(L:M)^2$$
$$\dagger=(y^4+1).\{(y-1)^2+y^2\}^2:\{(y+1)^2+y^2\}^2$$
$$=(y^4+1).\left\{\frac{(2y-1)^2+1}{2}\right\}^2:\left\{\frac{(2y+1)^2+1}{2}\right\}^2 \ldots(18),$$

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* *Bin-Aurifeuillian*, a name applied by the author to numbers of form $N=X^4+4Y^4$, see his Paper *On Aurifeuillians* in *Lond. Math. Soc. Proc.*, Vol. XXIX., 1898, pages 381—438.

† The colon (:) is used throughout this Paper as a *sign of multiplication* between the two large *algebraic factors* of any Bin-Aurifeuillian; thus, $x_2 = L : M$.

Complete factorisation of N depends now chiefly on the factorisability of the larger factor D (as the smaller factors L, M would generally be factorisable if D were), and can therefore be carried further than in Art. (6).

7*a*. *By the Factor-Tables.* These give *complete* factorisation of *all* such numbers (N) up to the limits $y_m = 54$ when y is *even*, and $y_m = 65$ when y is *odd* (Art. 4*b*),

$$(4.54^6 + 3.54^2)^2 = (54^4 + 1) . \{(53^2 + 54^2)(55^2 + 54^2)\}^2$$
$$\dagger = 8503057 . (25.229)^2 : (13.457)^2; \quad [22 \text{ figs.}]$$

$$(4.65^6 + 3.65^2)^2 = (65^4 + 1) . \{(64^2 + 65^2)(65^2 + 66^2)\}^2$$
$$\dagger = (2.8925313) . (53.157)^2 : (8581)^2; \quad [23 \text{ figs.}]$$

7*b*. *By the Factor-Table of* $(y^4 + 1)$, (Art. 4*e*). This gives *complete factorisation* (at sight) of *all* such numbers (N) up to the limits $y_m = 202$ (y *even*); $y_m = 249$ (y *odd*); [Art. 4*e*].

$$(4.202^6 + 3.202^2)^2 + 1 = (202^4 + 1) . \left(\frac{403^2 + 1}{2} : \frac{405^2 + 1}{2}\right)^2;$$
$$= (17.41.193.12377) . (5.109.149 : 82013)^2; \quad [29 \text{ figs.}]$$

$$(4.249^6 + 3.249^2)^2 + 1 = (249^4 + 1) . \left(\frac{497^2 + 1}{2} : \frac{499^2 + 1}{2}\right)^2;$$
$$= (2.41.241.194521).(5.17.1453 : 13.61.157)^2; \quad [30 \text{ figs.}]$$

It also gives *complete factorisation* (at sight) of many such numbers (N) up to the limit $y_m = 1000$ (Art. 4*e*),

$$(4.999^6 + 3.999^2)^2 + 1 = (999^4 + 1).\left(\frac{1997^2 + 1}{2} : \frac{1999^2 + 1}{2}\right)^2;$$
$$= (2.13.521.673.12569).(5.13.30677)^2 : (277.7213)^2; \quad [40 \text{ figs.}]$$

7*c*. *By the Congruence-Table*, $y^4 + 1 \equiv 0$, (Art. 4*f*). It has been explained (Art. 4*f*) that complete factorisation can often be obtained of $D = y^4 + 1$ up to limits far beyond those of the complete Table of $(y^4 + 1)$. When this has been *effected* for any high value of y, the complete factorisation of N depends on the factorisability of the two factors L, M; *i.e.* of $(Y^4 + 1)$ and $(Y'^4 + 1)$, where $Y = (2y - 1)$, $Y' = (2y + 1)$, both *odd* numbers: the limit y_m in the larger factor $Y' = (2y + 1)$ would be (Art. 4*a*, *b*, *c*),

By the Factor-Tables,	$y_m = 2121$.
By the $(y^2 + 1)$ Factorisation-Table (at sight),	$y_m = 7500$.
By the $y^2 + 1 \equiv 0$ Congruence Table,	$y_m = 23343$.

though, of course, by casting out divisors, the factorisation might in many cases be carried much further by aid of this last Table.

† See footnote † of page 171.

$$(4.2121^6+3.2121^2)^2+1=(2121^4+1).\left(\frac{4241^2+1}{2}:\frac{4243^2+1}{2}\right)^2$$
$$=(2.70001.144553441).(8993041:25.13.27697)^2;\ [44\text{ figs.}]$$

$$(4.7453^6+3.7453^2)^2+1=(7453^4+1).\left(\frac{14905^2+1}{2}:\frac{14907^2+1}{2}\right)^2$$
$$=(2.4857.15889.14160017)(17.37.176597:25.977.4549)^2;\ [48\text{ figs.}]$$

$$(4.22946^6+3.22946^2)^2+1=(22946^4+1).\left(\frac{45891^2+1}{2}:\frac{45893^2+1}{2}\right)^2$$
$$=(41.673.10729.25913.36137)(6217.169373:25.113.372773)^2;\ [56\text{ figs.}]$$

8. *Tables of* (y^r+1), &c. By choosing one or both of D, x_2 of form (y^r+1) the complete factorisation of which (for small values of $y=2, 3, 5$, &c.) may be taken from existing Tables, the formulæ of Art. 6, 7 will lead to *very high* factorisable numbers (N). An Abstract of the Tables, used for this purpose in the Articles which follow, is given below; the full Titles and a brief description are given in Appendix I.

1. *Lucas, E.*; 1879; $(2^{4n}+1)$ up to $4n=84$; $(2^{4n+2}+1)$ up to $4n+2=210$; [Many gaps].
2. *Bickmore, C. E.*; 1895; (a^n-1) up to $n=50$; ($a=2, 3, 5, 6, 7, 10, 11, 12$); [Many gaps].
3. *Cunningham, A.*; 1898; (2^n+1) up to $n=2\omega=210$; [L, M factors separate]; [Many gaps].
4. *Cunningham, A.*; [MS.]; (a^n+1) up to $n=50$; ($a=3, 5, 6, 7, 10, 11, 12$); [Many gaps].
5. *Cunningham, A. & Woodall, H. J.*; 1900; $(2^x.10^\alpha+1)$ up to $x=30$, $\alpha=10$; and some higher values. [Many gaps].

9. *Small Bases* (2, 3, &c.). Writing $y_1=y^r$ in Results (11), (12), (14),

$$y_1=y^r,\ x_1=1;\ D=y^{2r}+1;\ y_2=y^r(4y^{2r}+3),\ x_2=(4y^{2r}+1)\ldots(19),$$
$$\mathrm{N}=D.x_2^2=(4y^{3r}+3y^r)^2+1=(y^{2r}+1).(4y^{2r}+1)^2\ldots(20).$$

It is here supposed that $y^r\neq\square$ (*i.e.* r is *odd*): the case of $y^r=\square$ (or r *even*), being deferred to Art. 10.

9*a*. *Case of* $y=2$. The most interesting cases arise when $y=2$. All the numbers (N) of form

$$\mathrm{N}=(2^{3r+2}+3.2^r)^2+1=(2^{2r}+1).(2^{2r+2}+1)^2,\ [r\ odd.]\ldots\ldots\ldots.(21),$$

are completely factorisable—by aid of the Tables 1, 3 named in Art. 8—up to the high limit $r=41$ (except when $r=33$,

37, 39): many of these numbers (N) are *extremely large*, the highest being

$(2^{123}+3.2^{41})^2+1=(2^{82}+1).(2^{84}+1)^2$......[has 76 figures],

$=(5.181549.12112549$:* $10169.43249589).(\dagger 17; 241;$
$15790321; 3361.88959882481)^2$.

9*b*. *Cases of* $y=3, 5, 6,$ *&c.* Similar results follow from the use of the bases $y=3, 5, 6,$ &c.; but the factorisations cannot be carried nearly so high (as in case of $y=2$). The factorisation of $D=y^{2r}+1$ may be taken from the Tables 2, 4 named in Art. 8: those of $x_2=4y^{2r}+1$ are limited by the power of the congruence-table of $y^2+1\equiv 0 \pmod{p}$ [so that $2y^r \not> $ about 33000, see Art. 4*d*]. The highest numbers of this sort so factorisable at present are

$(4.3^{27}+3^{10})^2+1=(3^{18}+1).(4.3^{18}+1)^2=(2.5.73.530713).(397.3903481)^2$; [37 figs.]

$(4.5^{21}+3.5^7)^2+1=(5^{14}+1).(4.5^{14}+1)^2=(2.13.234750601).(89.641.427949)^2$;
[38 figs.]

$(4.6^{15}+3.6^5)^2+1=(6^{10}+1).(4.6^{10}+1)^2=(37.241.6781).(5.53.193.4729)^2$; [25 figs.]

$(4.7^{15}+3.7^5)^2+1=(7^{10}+1).(4.7^{10}+1)^2=(2.25.281.4021).(1129900997)^2$; [27 figs.]

[The small bases $y=10, 11, 12$ do not lead to such high numbers].

10. *Small Bases* (2, 3, &c.), [Aurifeuillian Extension]. Writing $y_1=y^{2r}$ in Results (11), (12), (14),

$$y_1=y^{2r},\ x_1=1;\ D=y^{4r}+1;\ y_2=y^{2r}(4y^{4r}+3),\ x_2=(4y^{4r}+1)\ldots(22),$$

$$N=(4y^{6r}+3y^{2r})^2+1=D.x_2^2=D.(L:M)^2$$

$$=(y^{4r}+1).[\{(y^r-1)^2+y^{2r}\}:\{(y^r+1)^2+y^{2r}\}]^2\ldots..(23),$$

$$=(y^{4r}+1).[\tfrac{1}{2}\{(2y^r-1)^2+1\}:\tfrac{1}{2}\{(2y^r+1)^2+1\}]^2\ldots(22a),$$

wherein the factor x_2 is now the Bin-Aurifeuillian $(4y^{4r}+1)$, as in Art. 7. Interesting cases arise by taking $r=2, 3, 5, 6, 7, 10, 11, 12$; the factorisations of D being taken from the Tables of $(y^{4r}+1)$ quoted in Art. 8, and those of L, M being taken from the Tables of $(2^{4r+2}+1)$ quoted in Art. 8, when $y=2$, or worked out from the large Factor-Tables and Congruence-tables (Art. 4*d*) in the other cases. Many of the numbers thus factorisable are extremely large.

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* The colon (:) is used as explained in Art. 7, footnote †, to separate the Aurifeuillian Factors (L, M).

† The semi-colons (;) are here used to separate the *algebraic* factors.

Examples.

$y = 2$; All N factorisable up to $r=21$ (except $r=17, 19, 20$);

$(2^{128}+3.2^{42})^2+1=(2^{84}+1)(2^{86}+1)^2$, [has 78 figures].

$=(17;241;15790321;3361.88959882481)(1759217765581:5.173.101653.500177)^2$

$y = 3$; All N factorisable up to $r = 6$;

$(4.3^{36}+3^{13})^2+1=(3^{24}+1)(4.3^{24}+1)^2$, [has 36 figures].

$=(2.17.193;\ 97.577.769)(25.42457:1064341)^2$.

$y = 5$; All N factorisable up to $r = 5$;

$(4.5^{30}+3.5^{10})^2+1=(5^{20}+1)(4.5^{20}+1)^2$, [has 44 figures].

$=(2.313;\ 241.632133361)(19525001:73.267637)^2$.

$y = 6$; All N factorisable up to $r = 4$;

$(4.6^{24}+3.6^{8})^2+1=(6^{16}+1)(4.6^{16}+1)^2$, [has 39 figures].

$=(353.1697.4709377)(3356641:25.29.4637)^2$.

$y = 7$; All N factorisable up to $r = 3$;

$(4.7^{18}+3.7^{6})^2+1=(7^{12}+1)(4.7^{12}+1)^2$, [has 32 figures].

$=(2.1201;\ 73.193.409)(234613:5.109.433)^2$.

$y = 10$; All N factorisable up to $r=4$;

$(4.10^{24}+3.10^{8})^2+1=(10^{16}+1)(4.10^{16}+1)^2$, [has 50 figures].

$=(353.449.641.1409.69857)(13.41.457.821:569.351529)^2$.

$y = 11$; All N factorisable up to $r = 3$;

$(4.11^{18}+3.11^{6})^2+1=(11^{12}+1)(4.11^{12}+1)^2$, [has 39 figures].

$=(2.7321;\ 10657.20113)(3540461:5.709157)^2$.

$y = 12$; All N factorisable up to $r = 3$;

$(4.12^{18}+3.12^{6})^2+1=(12^{12}+1)(4.12^{12}+1)^2$, [has 41 figures].

$=(89.233;\ 193.2227777)(17.109.3221:25.239017)^2$.

11. *Form* $y=2^\beta.10^\alpha$. Interesting results are obtained by writing $y=2^\beta.10^\alpha$ in the formulæ of Art. 5, 7; this gives

$$N=(2^{3\beta+2}.10^{3\alpha}+3.2^\beta.10^\alpha)^2+1=(2^{2\beta}.10^{2\alpha}+1)(2^{2\beta+2}.10^{2\alpha}+1)^2 \ldots (24).$$

Many of the factorisations required for the factors D, x, are given in Table No. 5 quoted in Art. 8. By that Table, together with MS. work since done by the author, *complete* factorisation of these numbers (N) has been obtained in the following cases

$\alpha =$	1;	2;	3;	4;
$\beta =$	1 to 9, 12, 13, 17;	1 to 6;	1, 2, 5, 8, 9;	1, 2, 7, 8;

Examples.

$(2^{53}.10^3+3.2^{17}.10)^2+1=(2^{34}.10^2+1).(2^{36}.10^2+1)^2$; [has 38 figures].

$=(13.41.53.653.93133).(89.21613:5730169)^2$;

$(2^{26}.10^{12}+3.2^{8}.10^{4})^2+1=(2^{16}.10^{8}+1)(2^{18}.10^{8}+1)^2$; [has 40 figures].

$=(17.17.113.337.641.929)(661.7741 : 5123201)^2$;

$(4.20^{18}+3.20^{6})^2+1=(20^{12}+1).(4.20^{12}+1)^2$; [has 49 figures].

$=(160001 ; 31177.821113)(41.3121561 : 17.7530353)^2$;

12. *Power-factors* (x_2^n, $n=2^r$). The general formula (14) of Art. 6 can be used *repeatedly* in such a way as to yield Pellian Numbers $N=Y^2+1$, containing the power-factor x_2^n, where $n=2^r$, but the factor D will now be raised as well as x_2. That formula (14) may be modified, by writing $2y=y_1$ therein, to the form (26*a*) below, which may then be *repeated* by simply changing y_1 into y_2, y_3, ... in succession.

$$y_1=y_1;\qquad N_1=y_1^2+1 \dots\dots (25),$$

$$y_2=\tfrac{1}{2}y_1(y_1^2+3);\ N_2=y_2^2+1=(\tfrac{1}{4}y_1^2+1).(y_1^2+1)^2=(\tfrac{1}{4}y_1^2+1).N_1^2 \dots (26a),$$

and the rest of the steps may now be derived *in succession* from the general formula

$$y_{r+1}=\tfrac{1}{2}y_r(y_r^2+3);\ N_{r+1}=y_{r+1}^2+1=(\tfrac{1}{4}y_r^2+1)(y_r^2+1)^2=(\tfrac{1}{4}y_r^2+1).N_r^2 \dots (26b),$$

or, using the abbreviation

$$N'_r=\tfrac{1}{4}y_r^2+1 \dots\dots (26c),$$

N_{r+1} may be written

$$N_{r+1}=N'_r\,.\,N_r^2 \dots\dots (26d).$$

and finally,

$$N_{r+1}=N'_r\,.\,N'^2_{r-1}\,.\,N'^4_{r-2}\dots N'^{2^{r-1}}_1\,.\,N_1^{2^r} \dots\dots (26e),$$

wherein all the factors are *accented*, except the last N_1. It will be seen that N_{r+1} contains $(1+2+2^2+\dots+2^r)=(2^{r+1}-1)$ *algebraic* factors, whereof the last N_1 enters in the 2^r*th* power. The process being a *direct* one, it is evident that numbers N_{r+1} may be evolved containing a *given* factor $N_1=(y_1^2+1)$ raised to any *given* power $n=2^r$, where $y_1\equiv 0 \pmod{2^r}$. But, observing that in the form (26*d*) all the power-factors enter into the term N_r, the complete factorisation of N_{r+1} depends really on its first term $N'_r=(\frac{1}{4}y_r^2+1)$, which also contains no algebraic factors, and is of magnitude $=\frac{1}{4}N_r$ very nearly.

[The power of factorisation of these numbers N_{r+1} is really not so great as with the numbers N of Art. 6, 7, because the form of y_r here is *not at choice*, but depends strictly on y_1: in fact complete factorisation is practically limited to the second step, $N_3=N'_2\,.\,N'^2_1\,.\,N_1^4$, wherein N_3 contains N_1^4].

Examples. These are here limited to $N_3 = y_3^2 + 1 = N'_2 . N'^2_1 . N_1^4$; where $N_1 = y_1^2 + 1$, and $y_1 = 4y'$.

1°. *By the Factor-Tables.* The limit is $y_1 = 20$; $N_1 = 20^2 + 1 = 401$;

$y_2 = 10(20^2+3) = 4030$; $N_2 = 4030^2 + 1 = (10^2+1)(20^2+1)^2 = 101.401^2$;

$y_3 = 2015.(4030^2+3)$; $N_3 = y_3^2 + 1 = (2015^2+1).(4030^2+1)^2$

$= (97.20929).(101^2.401^4)$; [22 figs.]

2°. *By the* (y^2+1) *Tables.* The limit is $y_1 = 36$; $N = 36^2 + 1 = 1297$;

$y_2 = 18.(36^2+3) = 23382$; $N_2 = y_2^2 + 1 = (18^2+1).(36^2+1)^2 = (25.13).(1297^2)$;

$y_3 = 11691.(23382^2+3)$; $N_3 = y_3^2 + 1 = (11691^2+1).(23382^2+1)^2$

$= (2.353.193597).(25.13)^2.1297^4$;

[26 figs.]

3°. *By the* (y^2+1) *Congruence-Tables.* The limit is $y_1 = 52$, $N_1 = 52^2 + 1 = 5.541$; but the result is more interesting when N_1 is a *prime.*

3°*a.* Take $y_1 = 40$, $N_1 = 40^2 + 1 = 1601$;

$y_2 = 20.(40^2+3) = 32060$; $N_2 = y_2^2 + 1 = (20^2+1).(40^2+1)^2 = 401.1601^2$;

$y_3 = 16030.(32060^2+3)$; $N_3 = y_3^2 + 1 = (16030^2+1).(32060^2+1)^2$;

$= 256960901.(401^2.1601^4)$; [27 figs.]

The limit could be carried much higher (than $y_1 = 52$) in certain cases by casting out the divisors shewn by the Congruence-Table.

13. *Second Pellian Form.* Referring to Result (13) of Art. 5,

$$N_3 = y_3^2 + 1 = (16y^5 + 20y^3 + 5y)^2 + 1 = D.x_3^2 = (y^2+1).(16y^4+12y^2+1)^2 \ldots (27),$$

and the factorisability of N depends chiefly on the larger factor x_3.

The large Factor-Tables give *complete* factorisation of *all* such numbers (N) up to $y = 27$ only; *e.g.*

$\{27(51^4+5.51^3+5)\}^2 + 1 = (27^2+1).(54^4+3.54^2+1)^2 = (2.5.73).(5.41\ 41521)^2$;

[17 figs.]

also, when $y = 5m \pm 2$, up to $y = 38$ (as in this case x_3 always contains 5): *e.g.*

$\{38(76^4+5.76^2+5)\}^2 + 1 = (38^2+1).(76^4+3.76^2+1)^2 = (5.17^2).(5.61.109441)^2$;

[19 figs.]

This factorisation could not be carried further (than for $y = 38$) without finding special divisors for x_3, which would be pretty laborious. As an aid in doing this, note that

$$x_3 = (4y^2+1)^2 + (2y)^2 = (6y^2+1)^2 - 5(2y^2)^2 = (4y^2-1)^2 + 5(2y)^2 \ldots (28),$$

Hence all prime divisors of x_3 must be of form $p = 20\varpi + 1, 9$.

14. *Use of Pellian Tables.* The published solutions (y, x) of the Pellian Tables (see Art. 14a) which extend continuously from $D = 2$ to 1500, may be used for giving factorisation *at sight* of Pellian Numbers $\text{N} = y^2 + 1 = D.x^2$. The peculiarity of these results is that the values of y, x frequently run *very high*, even for such *small* values of D, running in fact far beyond the powers of complete factorisation; *e.g.* $D = 1549$ gives $y > \frac{3}{2}.10^{35}$, $x > 10^{33}$. When x is within the power of the Factor-Tables (Art. 4), complete factorisation is easy; but when x is beyond their power, it becomes difficult.

Since x is a divisor of $\frac{1}{2}(y^2 + 1)$, it may be of form $x = \text{a}^2 + \text{b}^2$, and its factors (if any) must all be primes of form $p = 4\varpi + 1$, and that is practically all that is known about it.

14a. *Pellian Equation, Tables.* Here follows an Abstract of the published Tables of solutions of the two Pellian Equations

$$y^2 - Dx^2 = -1; \quad \eta^2 - D.\xi^2 = +1 \text{ ....} (1)$$

(so far as known to the present author), with brief indication of their scope, &c. Here x_0, y_0; ξ_0, η_0 denote minimum solutions. For full titles, and brief description, see Appendix II. (at end of this Paper).

No.	*Author*	*Date*	*D*	$y^2 - D.x^2 = -1$ [when possible]	$\eta^2 - D.\xi^2 = +1$ *Ref.*	*Solutions*	*Condition*
1.	*Legendre, A. M.*,	1798	$\neq \delta^2, \not> 1003$	Tab. xii; y_0, x_0	Tab. xii	η_0, ξ_0;	$y^2 - Dx^2 \neq -1$
2.	*Legendre, A. M.*,	1808	$\neq \delta^2, \not> 135$	Tab. x; y_0, x_0	Tab. x	η_0, ξ_0;	$y^2 - Dx^2 \neq -1$
3.	*Degen, C. F.*,	1817	All, $\not> 1000$	Tab. ii $\begin{cases} y_0, x_0 \\ D \neq \delta^2 + 1 \end{cases}$	Tab. i	η_0, ξ_0;	None
4.	*Legendre, A. M.*,	1830	$\neq \delta^2, \not> 1003$	Tab. x; y_0, x_0	Tab. x	η_0, ξ_0;	$y^2 - \text{D}x^2 \neq -1$
5.	*Richaud, H.*,	1887	1549	p. 182; y_0, x_0			
6.	*Bickmore, C. E.*,	1893	1001 to 1500	pp. 82-119; y_0, x_0	pp. 82-119	η_0, ξ_0;	$y^2 - Dx^2 \neq -1$
7.	*Cunningham, A. J. C.*,	1901	$\neq \delta^2, \not> 100$	p. 260; y_0, x_0	p. 260	η_0, ξ_0;	None
			$\neq \delta^2, \not> 20$	p. 261; Many y, x	p. 261	Many η, ξ;	None

15. *Table of Pellian Factorisations.* At the end of this Paper (page 185) follows a Table giving the elements (y, D, x) of the factorisations of *all* the High Pellian Numbers

$$\text{N} = y^2 + 1 = D.x^2 \text{ [with } y > 10^4 \text{ but } < 10^{17}] \text{ .....} (29).$$

contained in the Tables Nos. 1–7 of Art. 14a within the limits

$$\text{N} > 10^8, \text{ but } < 10^{34}, \text{ (given by } y > 10^4, \text{ but } < 10^{17}),$$

for which *complete factorisation* (of x) was within* the power of the large Factor-Tables after removal of all factors $\not> 1000$. The Table shews D and x resolved into their *prime* factors, giving thereby the *complete factorisation* of the High Pellian Numbers (N).

16. *Pellian Congruences* (mod x^2). By this term is here meant quadratic congruences of form

$$y^2 + 1 \equiv 0 \pmod{x^2} \quad\text{..................(30)}.$$

Every solution (y, x) of the Pellian Equation (1) gives *at sight* one root (y) of the above congruence, and the other root is $y' = x^2 - y$. Also, if x be *composite* (say $x = x_1 x_2 x_3 \ldots$) then y, y' are also roots of all the congruences

$$y^2 + 1 \equiv 0 \;(\text{mod } x_1^2,\, x_2^2,\, x_3^2,\, \&\text{c.}) \quad\text{.........(30}a\text{)},$$

but in this case y, y' would probably be $>$ some of the moduli x_1^2, x_2^2, &c.; they can be reduced by casting out any of the moduli as required.

The whole of the Factorisation-Table of Art. 15 can be used as a Table of solutions of the congruences (30, 30a).

The chief point of interest about these congruences is that the moduli are all *squares* (x^2, x_1^2, x_2^2, &c.). Now congruences with *square moduli* are otherwise somewhat difficult to solve,† whilst these solutions are given *at sight*.

17. *Pellian Congruences* (mod N^n, $n = 2^r$). The developments in Art. 12 give the means of finding one root (y_{r+1}) of the congruence

$$y_{r+1}^2 + 1 \equiv 0, \text{ mod } \mathrm{N}_1^n = (y_1^2 + 1)^n, \; [n = 2^r,\; y_1 \equiv 0 \pmod{2^r}] \ldots (31),$$

i.e. to a *given* modulus $\mathrm{N}_1 = (y_1^2 + 1)$, raised to a *given* power $n = 2^r$, when $y_1 \equiv 0 \pmod{2^r}$, in the most *direct* and simple manner. This problem is *easier* than that of Art. 12, in that the factorisation of the large Pellian Numbers N_{r+1} is not required, the *roots* (y_{r+1}) being the only sought quantities.

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* A few cases with $y > 10^8$ but $< 10^{12}$ (giving $N > 10^{17}$ but $< 10^{24}$), wherein complete factorisation of x was not thus obtainable, have been omitted.

† See the author's Paper *On Period-lengths of Circulates* in *Messenger of Mathematics*, Vol. XXIX., 1900, pp. 145–179, in which a systematic Method of doing this is developed; a Table of solutions of $y^2 + 1 \equiv 0 \pmod{p^2}$ is given for prime moduli p up to $p = 199$, on p. 166; also Examples for many higher prime moduli on pp. 162, 163. The Method in the present Paper has enabled the Table on p. 166 to be completed for all prime moduli $p \not> 1000$, and for many four figure primes. It is hoped to publish this hereafter.

The quantities $(y_2, y_3, \ldots y_{r+1})$ of Results (26*a*–*b*) are in fact the sought roots of the successive congruences

$$y_2^2+1\equiv 0(\text{mod } N_1^2);\ y_3^2+1\equiv 0(\text{mod} N_1^4);\ldots y^2_{r+1}+1\equiv 0(\text{mod } N_1 2^r)\ldots(31a)$$

and it is only necessary to form the successive roots $y_2, y_3, \ldots, y_{r+1}$, by the Rules (26*a*, *b*).

Of course at every step the roots $y_1, y_2, \ldots, y_r$ must be *even*; this is secured by the condition $y_1 \equiv 0 \pmod{2^r}$, and the process stops at the step when y_{r+p} becomes *odd*. This process has the inconvenience that at each step (after y_2) the root y_{r+1} may be > the modulus N_1^n, and the ratio of excess $y_{r+1} : N_1^n$ increases at each step; so that reduction is required for each congruence after all the steps are complete. On the other hand, it must be observed that congruences to modulus N_1^4, N_1^8, &c., are generally difficult to solve.

Examples. The results are interesting when the modulus $N_1 =$ a Fermat's *prime*.

Write $E_x = 2^{2^x}$, $F_x = 2^{2^x}+1 = E_x + 1$, a Fermat's *prime*; here $E_{x+1} = E_x^2$.

Take $y_1 = E_{x-1}$; then $N_1 = y_1^2 + 1 = F_x$.

$y_2 = \frac{1}{2}E_{x-1}(E_x+3)$; $y_2^2+1\equiv 0(\text{mod } F_x)$, and $y_2 < F_x^2$ *always*.

$y_3 = \frac{1}{4}E_{x-1}(E_x+3).\{\frac{1}{4}E_x(E_{x+1}+6E_x+9)+3\}$; $y_3^2+1\equiv 0, (\text{mod } F_x^4)$.

Here $y_3 < F_x^4$, when $F_x = 17$, and $y_3 > F_x^4$ when $F_x > 17$.

1°. Take $x=2$; then $y_1 = 4$, $N_1 = 17 = F_2$.

$y_2 = 38$, $y_3 = 27493$ are the roots to mod 17^2 and 17^4.

2°. Take $x=3$; then $y_1 = 16$, $N_1 = 257 = F_3$.

$y_2 = 2072$, $y_3 = 1036.4293187 > F_3^4$, which requires reduction, giving

$$85271331^2+1\equiv 0, (\text{mod } F_3^4).$$

3°. Take $x=4$; then $y_1 = 256$, $N_1 = 65537 = F_4$.

$y_2 = 128.65539 = 8388992$; $y_3 = 64.65539(2^{14}.65539^2+3) > F_4^4$.

Appendix I.

Here follows a brief description of the *Factorisation-Tables* of $N = y^n + 1$ (*n even*), referred to in Art. 8 (so far as used in this Paper), numbered as in the Abstract in that Article; with full titles, and dates, of the Works wherein they will be found.

The abbreviation A. P. F. is used in what follows for "Max. Algebraic Prime Factor."

1. Lucas, Ed. *Sur la série récurrente de Fermat*, Rome, 1879.

The two short Tables, pp. 9, 10, give all the *specific* divisors of the A.P.F. of $N=(2^m+1)$, when $m=4n+2$ or $4n$, for the following values of m,

$$m=4n+2=2 \text{ to } 102,\ 110,\ 114,\ 126,\ 130,\ 138,\ 150,\ 210;$$

$$m=4n \quad =4 \text{ to } 60,\ 72,\ 84;$$

and gives also incomplete sets for other values of m.

Specific divisors of (a^m+1) are those primes which divide into (a^m+1), but not into any sub-multiple $(a^\mu+1)$ of (a^m+1); m being composite and μ one of its factors. Thus a complete set of the specific divisors of (a^m+1) is also a *complete factorisation* of the A.P.F. of (a^m+1) in all cases except when m is a multiple of μq, where q is a specific divisor of $(a^\mu+1)$: in this (exceptional) case q is a factor of the A.P.F of (a^m+1), but would be omitted by Lucas's rule from the set of divisors of (a^m+1), as not being a specific divisor thereof.

Ex. 5 is omitted from the printed factors of the A.P F. of both $(2^{10}+1)$, and $(2^{50}+1)$, because $(2^2+1)=5$.

13 is omitted from the printed factors of the A.P.F. $(2^{78}+1)$, because (2^6+1) contains 13.

This omission of factors of the A.P.F. of (2^m+1) when *not specific divisors* of (2^m+1) is a decided inconvenience when a complete set of factors is wanted.

Erratum in above Tables. Insert $p=12112549$ as a divisor of $(2^{82}+1)$.

2. Bickmore, Chas. E. *On the numerical factors of* (a^n-1), in *Messenger of Mathematics*, Vol. xxv, 1896, pp. 1-44.

The Table on pp. 43, 44 shews all the prime factors (so far as known to the author) of the A.P.F. of (a^n-1), with $n=1$ to 50, when $a=2, 3, 5, 6, 7, 10, 11, 12$.

When any prime factor of (a^n-1) is contained in some of the algebraic sub-multiples of (a^n-1), as well as in (a^n-1), then that factor is shewn *raised to the full power* in which it enters into the complete (a^n-1).

Ex. 3 is a divisor of (2^1+1), $(2^3+1)\div(2^1+1)$, and of $(2^9+1)\div(2^3+1)$, which are the A.P.F. of (2^2-1), (2^6-1), $(2^{18}-1)$. Accordingly 3^2, 3^3 are printed (by the above Rule) among the factors of (2^6-1), and $(2^{18}-1)$, although they are not divisors of the A.P.F. of (2^6-1), $(2^{18}-1)$ respectively. This results in a *redundance* of divisors printed as (apparently) divisors of the A.P.F., and is a decided inconvenience when the divisors of the A.P.F. are wanted. By Lucas's Rule, such divisors would be omitted altogether, (not being specific divisors).

This Table is a great advance on any previous Table for bases >2. It contains many gaps, and some Errata. A further Table on pp. 37, 38 of the *Messenger of Mathematics*, Vol. xxvi, 1897, contains a long List of Addenda (which fill up many of the gaps) and some Errata for the original Table.

3. Cunningham, Allan; Lt.-Col., R.E. *On Aurifeuillians*, in *Proc. Lond. Math. Soc.*, Vol. xxix., pp. 381-438.

The Table on pp. 406, 407, shows the *complete factorisation* (into prime factors) of the A.P.F. of the Bin-Aurifeuillian $N=(2^n+1)$, where $n=2\omega$, for

$$n=2\omega=2 \text{ to } 102,\ 110,\ 114,\ 126,\ 130,\ 138,\ 150,\ 210;$$

and also partial factorisation for several other values of n.

This Table is a *recomputed* edition of this part of Lucas's Table (No. 1) above, with the advantage that the "Aurifeuillian Factors" (L, M) are shewn separate, and that *all* the factors (not merely the specific factors) of the A.P.F. are shewn; [the omission in Lucas's Table No. 1, and the redundance in Bickmore's Table No. 2, are thus avoided].

Erratum in above Table, p. 407; The factor M of $(2^{90}+1)$ should be 29247661.

4. Cunningham, Allan. This Table, being at present only in MS., need not be described.

5. Cunningham, Allan, and Woodall, H. J. *Factorisation of* $(2^x.10^a \mp 1)$, pub. in *Math. Questions with their Solutions from the Educational Times*, Vol. 73, 1900, pp. 83—91. This gives the complete factorisation of $N = (2^x.10^a \mp 1)$ up to $x = 30$ and $a = 10$, and also for a good many higher values. There are of course many gaps: some of these have been since filled up in MS.

Appendix II.

Here follows a brief description of the Pellian Equation Tables referred to in Art. 14a, numbered as in the Abstract of that Article, with the full titles, &c., of the Works wherein they will be found.

1. Legendre, A. M. *Essai sur la Théorie des Nombres*, Paris, 1798.

Table XII. (at the end of the volume) contains (in twelve 4to pages) the *minimum solutions* (m, n) of *one or other* of the two Pellian Equations; viz.

Of $m^2 - a.n^2 = -1$ (when possible); Of $m^2 - a.n^2 = +1$ (in other cases) for *all* (non-square) values of a from 2 to 1003. The solutions (m, n) are printed in the form of a fraction $\left(\frac{m}{n}\right)$ in an inconveniently small (but fairly clear) type.

This Table has the inconvenience that the solutions of the two equations are printed together *in one Table*, the Argument a running continuously (but omitting squares) from 2 to 1003, without any mark to shew to which equation the solution (m, n) belongs. The two cases can be readily distinguished by examining the units digits, say m_0, a_0, n_0 of m, a, n; for

$$m^2 - a\,n^2 = \mp 1, \text{ according as } m_0^2 - a_0\,n_0^2 \equiv \mp 1 \pmod{10}.$$

The necessity of this preliminary examination of every case (when $a = 4n + 1$), and the omission of the solution of the equation $m^2 - a.n^2 = +1$, when that of $m^2 - an^2 = -1$ is given, are decided inconveniences in this Table.

Errata. There are Errata in the values of either, or both of, m, n for the 38 values of a shewn below:

a = 133, 214, 236, 301, 307, 331, 343, 344, 355, 365, 397, 501, 526,
Err. in m; m, n; m; n, m, m; m, n; n, n, m, m, n, m, m, n;

a = 533, 613, 619, 629, 655, 664, 671, 694, 718, 732, 753, 771, 801,
Err. in m, m; m, n; m; m, n; m, m; m, n; m, m, n, n; m, n; m, n;

a = 806, 809, 851, 856, 865, 871, 878, 886, 914, 965, 995, 1001;
Err. in m, m, n; m; m, n; m; m, n; m, n; n, m, m, n; m, m.

The above Abstract will suffice to *fix the position* of the Errata; it has not been thought necessary to give the actual Corrigenda, as this (old) Edition is now practically superseded by the newer (3rd) Edition (quoted below); to which reference should be made.

2. Legendre, A. M. *Essai sur la Théorie des Nombres,* 2nd Edition, Paris, 1808.

Tab. X. (at the end of the Text) contains (on a single 4to page) the *minimum solutions* (m, n), written in the form of a fraction $\left(\frac{m}{n}\right)$ of *one or other* of the two Pellian Equations, viz.

Of $m^2 - A.n^2 = -1$, (when possible); Of $m^2 - A.n^2 = +1$, (in other cases) for all (non-square) values of A from 2 to 135.

This Table is merely a *reprint* of part of the similar Table in the previous Edition (described under No. 1 above), but the type is clearer. It has the same inconveniences as in that Table (see No. 1 above). There appear to be *no Errata.*

3. Degen, C. F. *Canon Pellianus,* &c. Hafniæ, 1817.

This book contains the *minimum solutions* (y, x) of *both* the Pellian Equations as follows (besides the elements of the continued fraction development of $\sqrt{(a)}$ which lead to them);

Tab. I. pp. 1 to 106. Of $y^2 - ax^2 = +1$, for all values of $a = 1$ to 1000.

Tab. II. pp. 109 to 112. Of $y^2 - ax^2 = -1$, for all values of a when such solution is possible (but omitting the case of $a = \delta^2 + 1$) from $a = 1$ to 1000.

Corrigenda (in solutions y, x), [Argument a].

p. 17; $a = 238$; For $x = 1756$, Read* $x = 756$.

p. 36; $a = 437$; For $y = 4499$, Read‡ $y = 4599$.

p. 65; $a = 672$; For $y = 327$, Read‡ $y = 337$.

p. 75; $a = 751$; The last 7 figures of y should be †...4418960.

p. 84; $a = 823$; Insert 47† after 235170 in value of y.

p. 96; $a = 919$; The last 14 figures of x should be ‡...36759781499589.

p. 100; $a = 945$; For $y = 27551$, Read‡ $y = 275561$.

p. 100; $a = 951$; For $y = 22420806$, Read‡ $y = 224208076$.

p. 107; The Title of Tab. II. states that the solutions of $y^2 - ax^2 = -1$ are given whenever possible; but all cases of $a = \delta^2 + 1$ are actually *omitted.*

4. Legendre, A. M. *Théorie des Nombres,* Vol. i, 3rd Ed., Paris, 1830.

Tab. X. (on the eight 4to pages at the end of the volume) contains the *minimum solutions* (x, y) of one or other of the two Pellian Equations, viz.

Of $x^2 - N.y^2 = -1$, (when possible); Of $x^2 - N.y^2 = +1$ in other cases for all (non-square) values of N from 2 to 1003.

This Table is nearly the same as that in the first Edition (see No. 1 above); but is more correct (most of the Errata having been corrected). The values of x, y are printed as a ratio $(x:y)$, instead of as a fraction as in the original. The Arguments N are not quite consecutive, as in the

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* This Erratum is given on p. 112 of Degen's Work.

† These two Errata are taken from *Journ. de Mathém. Elément,* 1887, p. 183; *Correspondence de H. Richaud.*

‡ These five Errata were discovered by the present writer.

original: those cases which have very large values of x, y are placed in a separate Table at the end (so as to save space in printing). This Table is otherwise like the original, and has the *same inconveniences* (see No. 1 above). Note that the x, y of this Table are the y, x respectively of Degen's Table (No. 3 above), and the N of this Table is the a, A of the previous Tables.

Corrigenda† [Argument N].

$N =$ 94, Read $x = 2143295$	$N = 271$, x should end with ...983600
,, 116, ,, $x = 9801$	,, 749, x ,, ,, ,, ...84895
,, 149, ,, $y = 9305$	,, 751, x ,, ,, ,, ...424418960
,, 308, ,, $x = 351$	,, 823, Insert 47 after 235170 in x
,, 479, ,, $y = 136591$	,, 809, x should begin with 43385...
,, 629, ,, $x = 7850$	
,, 667, ,, $y = 4147668$	

5. Richaud, H. The *Correspondance de H. Richaud* in *Journ. de Mathém. Element*, 1887, p. 182, contains the lowest solution (y_0, x_0) of $y^2 - 1549x^2 = -1$. Here $y_0 > \frac{3}{2} \cdot 10^{35}$, $x_0 > 3 \cdot 10^{33}$.

6. Bickmore, Chas. E. *Tables connected with the Pellian Equation, &c.*, in *Brit. Assoc. Report* of 1893, pp. 73—120. [The Text is by A. Cayley].

The Table, pp. 82 to 119, is a continuation of Degen's Table (No. 3 above), with slight changes. It gives the *minimum solution* of *one or other* of the two Pellian Equations (besides the elements of the continued fraction development of $\sqrt{a}$ which lead to them) as follows,

Of $y^2 - ax^2 = -1$ (when possible); Of $y^2 - ax^2 = +1$ in other cases

for all (non-square) values of a from 1001 to 1500.

The solutions (y, x) of the two Equations are printed together *in one Table*, the Argument (a) running continuously (but omitting squares) from 1001 to 1500: but the two cases are distinguished by the Argument (a) being marked with an asterisk (*) in the first case, and being left unmarked in the second case.

The omission of the solution of the important equation $y^2 - ax^2 = +1$, whenever $y^2 - ax^2 = -1$ is possible, is a decided inconvenience.

Note one *Corrigendum*, page 109. The Argument $a = 1361$ should be 1361*.

7. Cunningham, Allan; Lt.-Col., R. E. *Quadratic Partitions*, London, 1904.

(1) The Table on p. 260 contains the *minimum solutions* (τ, υ) of *both* the Pellian Equations, as follows:—

Of $\tau^2 - D.\upsilon^2 = +1$, for *all* (non-square) values of $D < 100$.

Of $\tau^2 - D.\upsilon^2 = -1$, for *all* (non-square) values of $D < 100$ (when possible).

(2) The Table on p. 261 contains multiple solutions (τ_0, υ_0), (τ_1, υ_1), (τ_2, υ_2), &c. of *both* the Pellian Equations, viz. of $\tau^2 - D.\upsilon^2 = -1$ (when possible), and of $\tau^2 - D.\upsilon^2 = +1$, for *all* (non-square) values of D from $D = 2$ to 20.

† Most of these Corrigenda will be found in *Journ. de Mathém. Elements*, 1886, p. 139; and 1887, p. 183; under Correspondence de M. H. Richaud; and in *Atti dell' Accad. Pont. d. Nuovi Lincei*, t. XX., Rome, 1866, in a Note by M. E. Catalan.

High Pellian Factorisations (N).

$N = (y^2 + 1) = D . x^2$, [resolved into prime factors]; see Art. 15.

y	*D*	*x*	*y*	*D*	*x*
11,782	701	5·89	3,940,598	5	89·19801
12,238	5	13·421	4,832,118	157	5·13·17·349
14,942	5·193	13·37	5,534,843	2·373	5·40529
17,684	17	4289	6,406,803	2·5	17·37·3221
18,018	1093	5·109	8,118,568	857	25·11093
20,457	2·13·29	5·149	8,890,182	109	25·34061
21,490	1229	613	9,369,319	2	37·179057
23,156	233	37·41	14,752,278	5·113	13·47741
23,382	13	5·1297	15,489,282	829	5·17·6329
25,382	1373	5·137	18,245,310	709	13·52709
28,488	5·13·17	857	21,019,276	1097	13·48817
29,718	61	5·761	24,314,110	461	17·29·2297
29,851	2·661	821	24,715,982	797	5·13·13469
30,235	2·397	29·37	27,628,256	1217	791969
38,899	2·709	1033	30,349,818	13	25·109·3089
42,801	2·13·53	1153	33,995,032	17·73	5·17·11353
45,368	29·37	5·277	41,009,716	617	17·97117
47,321	2	33461	54,608,393	2	5·13·13·45697
69,051	2·269	13·229	54,610,269	2·389	17·41·53·53
71,264	353	3793	70,600,734	1213	2027117
71,847	2·5·61	2909	70,711,162	5	233·135721
84,906	997	2689	71,011,068	241	25·182969
104,092	5·277	2797	87,050,499	2·509	433·6301
113,582	149	5·1861	99,484,332	1489	5·233·2213
114,669	2·17·37	53·61	126,862,368	313	5·17·29·2909
168,717	2·5	53353	128,377,240	521	733·7673
174,293	2·277	5·1481	153,352,043	2·733	5·801037
217,318	1301	25·241	189,471,332	449	5·1788341
219,602	5	17·53·109	218,623,878	5·137	13·29·22157
241,326	877	29·281	348,345,108	5·173	233·50833
275,807	2	25·29·269	393,166,618	1013	5·29·85193
328,173	2·5·97	41·257	395,727,950	509	41·427813
348,711	2·569	10337	419,288,307	2·461	25·552341
352,618	317	5·17·233	731,069,390	941	17·37·37889
409,557	2·149	25·13·73	854,992,268	1193	5·1097·4513
600,632	593	5·4933	1,111,225,770	181	13·17·97·3853
683,982	17·29	5·61·101	2,291,286,382	653	5·13·1379461
964,140	13·37	43961	2,746,864,744	953	449·198173
1,063,532	281	5·12689	2,894,863,832	569	5·17·1427753
1,166,876	17	283009	2,959,961,778	5·233	53·797·2053
1,262,101	2·541	17·37·61	4,115,086,707	2·293	5·61·349·1597
1,343,018	773	5·9661	5,767,329,724	1433	701·217337
1,369,326	757	157·317	6,547,100,182	1493	25·61·111109
1,607,521	2	137·8297	7,376,748,868	977	5·13·3630817
1,764,132	193	5·109·233	8,920,484,118	277	5·157·682777
2,086,882	17·61	5·13·997	20,478,302,982	397	5·17·37·173·1889
3,375,918	1429	5·53·337	106,316,171,432	881	25·173·389·2129
3,434,907	2·13·41	5·53·397	930,015,700,509	2·701	113·181·1214393
3,881,493	2·29	5·13·7841			

NOTE ON AN EXPANSION OF $(1+x)^k$ IN LEGENDRIAN COEFFICIENTS.

By *J. W. L. Glaisher*.

§ 1. It was shown by Bauer in Vol. LVI. (p. 114) of *Crelle's Journal** that

$$\left(\frac{1+x}{2}\right)^k = \frac{1}{k+1} P_0(x) + 3\,\frac{k}{(k+1)(k+2)} P_1(x)$$
$$+ 5\,\frac{k(k-1)}{(k+1)(k+2)(k+3)} P_2(x) + \&c.,$$

where $P_0(x) = 1$ and $P_n(x)$ denotes the n^{th} Legendrian coefficient of x.

This expansion is true for all values of k for which the series is convergent.

§ 2. If k is a positive integer the series terminates; but if k is the half of a positive uneven integer a curious change occurs in the form of the series after the r^{th} term. The object of this note is to draw attention to this change.

Putting $k = r - \frac{1}{2}$, r being a positive integer, and writing for brevity P_n for $P_n(x)$, the expansion is

$$\frac{(1+x)^{r-\frac{1}{2}}}{2^{r+\frac{1}{2}}} = \frac{P_0}{2r+1} + 3\,\frac{2r-1}{(2r+1)(2r+3)} P_1$$
$$+ 5\,\frac{(2r-1)(2r-3)}{(2r+1)(2r+3)(2r+5)} P_2 + \ldots$$
$$+ (2r-1)\,\frac{(2r-1)(2r-3)\ldots 3}{(2r+1)(2r+3)\ldots(4r-1)} P_{r-1}$$
$$+ 1^2.3^2.5^2\ldots(2r-1)^2 \left\{ \frac{2r+1}{1.3.5\ldots(4r+1)} P_r - \frac{2r+3}{3.5.7\ldots(4r+3)} P_{r+1} + \ldots \right.$$
$$\left. + (-1)^s \frac{2r+2s+1}{(2s+1)(2s+3)\ldots(4r+2s+1)} P_{r+s} + \&c. \right\}.$$

In the first series the terms are all positive and negative. In the second they are alternately positive and negative, and in each term the number which occurs in the numerator is the same as the middle factor of the denominator.

Thus the term in P_n in the alternating series is

$$1^2.3^2\ldots(2r-1)^2.(-1)^{n-r}\,\frac{2n+1}{(2n-2r+1)(2n-2r+3)\ldots(2n+2r+1)} P_n,$$

* "Von den coefficienten der Reihen von Kugelfunctionen einer Variablen," pp. 101–121.

that is

$$1^2.3^2\dots(2r-1)^2.(-1)^{n-r}\frac{1}{(m^2-2^2)(m^2-4^2)\dots\{m^2-(2r)^2\}}P_n,$$

where $m=2n+1$.

The alternating series may therefore be written

$$1^2.3^2\dots(2r-1)^2\,\Sigma_{n=r}^{n=\infty}\,(-1)^{n-r}\frac{P_n}{(m^2-2^2)(m^2-4^2)\dots\{m^2-(2r)^2\}},$$

and the whole expansion may be expressed by the formula

$$\frac{(1+x)^{r-\frac{1}{2}}}{2^{r+\frac{1}{2}}}=\Sigma_{n=0}^{n=r-1}(2n+1)\frac{(2r-1)(2r-3)\dots(2r-2n+1)}{(2r+1)(2r+3)\dots(2r+2n+1)}P_n$$

$$+(-1)^r1^2.3^2\dots(2r-1)^2\,\Sigma_{n=r}^{n=\infty}\,(-1)^n\frac{P_n}{(m^2-2^2)(m^2-4^2)\dots\{m^2-(2r)^2\}},$$

where, corresponding to $n=0$, the term is $\frac{1}{2r+1}$, and m denotes $2n+1$.

§ 3. The general term of the first series necessarily includes the terms of the second series, for the preceding formula has been derived from

$$\frac{(1+x)^{r-\frac{1}{2}}}{2^{r+\frac{1}{2}}}=\Sigma_{n=0}^{n=\infty}\,(2n+1)\frac{(2r-1)(2r-3)\dots(2r-2n+1)}{(2r+1)(2r+3)\dots(2r+2n+1)}P_n;$$

and it will now be shown that the general term of the second series also includes the terms of the first.

For, if $r<n$,

$$(-1)^{n+r}\frac{1^2.3^2\dots(2r-1)^2}{(m^2-2^2)(m^2-4^2)\dots\{m^2-(2r)^2\}}$$

$$=(-1)^{n+r}\frac{1^2.3^2\dots(2r-1)^2}{-1.-3\dots-(2r-2n-1)\times1.3\dots(2n-1)\times(2n+3)\dots(2n+2r+1)}$$

$$=(-1)^{n+r}(2n+1)\frac{1^2.3^2\dots(2r-1)^2}{(-1)^{r-n}1.3\dots(2r-2n-1)\times1.3\dots(2n+2r+1)}$$

$$=(2n+1)\frac{1.3\dots(2r-1)}{1.3\dots(2r-2n-1)\times(2r+1)(2r+3)\dots(2r+2n+1)}$$

$$=(2n+1)\frac{(2r-2n+1)(2r-2n+3)\dots(2r-1)}{(2r+1)(2r+3)\dots(2r+2n+1)},$$

which is the coefficient of P_n in the first series.

Thus we may write the expansion also in the form

$$\frac{(1+x)^{r-\frac{1}{2}}}{2^{r-\frac{1}{2}}}=(-1)^r1^2.3^2\dots(2r-1)^2\,\Sigma_{n=0}^{n=\infty}\,(-1)^n\frac{P_n}{(m^2-2^2)\dots\{m^2-(2r)^2\}}.$$

§ 4. Integrating between the limits − 1 and 1 we have also two forms for $\int_{-1}^{1}(1+x)^{r-\frac{1}{2}}P_n(x)\,dx$, viz.

$$\int_{-1}^{1}(1+x)^{r-\frac{1}{2}}P_n(x)\,dx=2^{r+\frac{1}{2}}\frac{(2r-1)\,(2r-3)\dots(2r-2n+1)}{(2r+1)\,(2r+3)\dots(2r+2n+1)},$$

and
$$=(-1)^{r+n}\,2^{r+\frac{3}{2}}\frac{1^2.3^2\dots(2r-1)^2}{m\,(m^2-2^2)\,(m^2-4^2)\dots\{m^2-(2r)^2\}},$$

or, changing the sign of x,

$$\int_{-1}^{1}(1-x)^{r-\frac{1}{2}}P_n(x)\,dx=2^{r+\frac{1}{2}}\frac{(2r-1)\,(2r-3)\dots(2r-2n+1)}{(2r+1)\,(2r+3)\dots(2r+2n+1)}$$

$$=(-1)^{r}\,2^{r+\frac{3}{2}}\frac{1^2.3^2\dots(2r-1)^2}{m\,(m^2-2^2)\,(m^2-4^2)\dots\{m^2-(2r)^2\}}.$$

§ 5. From this last formula we may deduce the value of a somewhat complicated definite integral which is perhaps worth noticing.

Transforming this formula by taking $x=1-2k^2$, we have

$$\int_0^1 k^{2r}P_n(1-2k^2)\,dk=(-1)^r\frac{1^2.3^2\dots(2r-1)^2}{m\,(m^2-2^2)\,(m^2-4^2)\dots\{m^2-(2r)^2\}}.$$

Now, p being unrestricted,

$$\cos(p\sin^{-1}x)=1-\frac{p^2}{2!}x^2+\frac{p^2(p^2-2^2)}{4!}x^4-\frac{p^2(p^2-2^2)(p^2-4^2)}{6!}x^6+\&c.,$$

whence, putting kx for x,

$$\frac{1-\cos(p\sin^{-1}kx)}{k^2}=\frac{p^2}{2!}x^2-\frac{p^2(p^2-2^2)}{4!}k^2x^4+\frac{p^2(p^2-2^2)(p^2-4^2)}{6!}k^4x^6-\&c.$$

Multiplying throughout by $P_n(1-2k^2)$, replacing p by $m=2n+1$, and integrating with respect to k between the limits 0 and 1, we find

$$\int_0^1\frac{1-\cos(m\sin^{-1}kx)}{k^2}P_n(1-2k^2)\,dk$$

$$=m\left\{\frac{1}{2!}x^2+\frac{1}{4!}x^4+\frac{1^2.3^2}{6!}x^6+\frac{1^2.3^2.5^2}{8!}x^8+\&c.\right\}$$

$$=m\,\{x\sin^{-1}x+\sqrt{(1-x^2)}\},$$

that is

$$\int_0^1\left[\frac{\sin\{(n+\frac{1}{2})\sin^{-1}kx\}}{k}\right]^2P_n(1-2k^2)\,dk=(n+\tfrac{1}{2})\{x\sin^{-1}x+\sqrt{(1-x^2)}\},$$

and therefore also, differentiating,

$$\int_0^1 \frac{\sin\{(2n+1)\sin^{-1}kx\}}{k\sqrt{(1-k^2x^2)}} P_n(1-2k^2)\,dk = \sin^{-1}x.$$

§ 6. In the particular case of $x=1$ these formulæ give (on putting $k=\sin\theta$)

$$\int_0^{\frac{1}{2}\pi} \left\{\frac{\sin(n+\frac{1}{2})\theta}{\sin\theta}\right\}^2 P_n(\cos 2\theta)\cos\theta\,d\theta = \tfrac{1}{4}(2n+1)\pi,$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin(2n+1)\theta}{\sin\theta} P_n(\cos 2\theta)\,d\theta = \tfrac{1}{2}\pi.$$

The latter formula may be easily verified; for, replacing θ by $\frac{1}{2}\theta$, it becomes

$$\int_0^{\pi} \frac{\sin\frac{1}{2}(2n+1)\theta}{\sin\frac{1}{2}\theta} P_n(\cos\theta)\,d\theta = \pi,$$

that is

$$\int_0^{\pi} (1+2\cos\theta+2\cos 2\theta+\ldots+2\cos n\theta)\,P_n(\cos\theta)\,d\theta = \pi.$$

Now $\quad P_n(\cos\theta) = A_0 + A_2\cos 2\theta + \ldots + A_n\cos n\theta,$

or $\qquad A_1\cos\theta + A_3\cos 3\theta + \ldots + A_n\cos n\theta,$

according as n is even or uneven. In the former case the integral

$$= (A_0 + A_2 + \ldots + A_n)\,\pi,$$

and in the latter

$$(A_1 + A_3 + \ldots + A_n)\,\pi,$$

viz. in both cases it

$$= P_n(1)\,\pi = \pi.$$

§ 7. The present note was suggested by §§ 17–26 (pp. 245–251) of a paper* in Vol. XXXVII. of the *Quarterly Journal.* In § 28 of that paper an independent proof of the formula

$$\int_{-1}^1 (1-x)^p P_n(x)\,dx = (-1)^n\,2^{p+1}\frac{p(p-1)\ldots(p+n+1)}{(p+1)(p+2)\ldots(p+n+1)}$$

is given.

* "On the expansions of $\int_0^1 k^n F(\phi)\,dk$ and $\int_0^1 k^n E(\phi)\,d\phi$, $F(\phi)$ and $E(\phi)$ being the Legendrian elliptic integrals," pp. 235–276.

ON THE FIGURE CONSISTING OF A REGULAR PENTAGON AND THE LINE AT INFINITY.

By *W. Burnside.*

LET $A_1B_1C_1D_1E_1$ be a regular pentagon. Denote by A_2, B_2, C_2, D_2, E_2, the points of intersection of B_1C_1, D_1E_1; C_1D_1, E_1A_1; D_1E_1, A_1B_1; E_1A_1, B_1C_1; A_1B_1, C_1D_1; respectively; and by A_0, B_0, C_0, D_0, E_0 the points at infinity on C_1D_1, D_1E_1, E_1A_1, A_1B_1, B_1C_1. The 15 points to which letters have been assigned are the complete intersections of the 6 lines consisting of the sides of the pentagon and the line at infinity.

If Q is any point of a plane in which O is a given point and o a given line; and if on OQ, meeting o in O', Q' is taken so that $OQO'Q'$ is harmonic, then the projective transformation which changes Q into Q' is called a perspective of order two; and O, o are the fixed point and the fixed line of the perspective.

A mere inspection of the figure consisting of the sides of the pentagon and the line at infinity, taking account of the obvious property that A_1A_2 is bisected by C_1D_1, shews that it is transformed into itself by a set of 15 perspectives of order 2. That the perspective of which A_0 is the fixed point and A_1A_2 the fixed line transforms the figure into itself, follows from the figure being symmetrical with respect to A_1A_2. The perspective of which A_1 is the fixed point and A_2A_0 the fixed line, leaves A_1B_1 and A_1E_1 unchanged and interchanges C_1D_1 with the line at infinity. Further it changes B_0 and E_0 into C_1 and D_1 respectively, and leaves A_2 unchanged. Hence it interchanges B_1C_1 and D_1E_1. Similarly, the perspective of which A_2 is the fixed point and A_1A_0 the fixed line leaves B_1C_1 and D_1E_1 unaltered, and interchanges C_1D_1 with the line at infinity. It also changes C_0 and D_0 into E_2 and B_2 respectively, and leaves A_1 unchanged. Hence it interchanges A_1B_1 and A_1E_1.

The triangle (in the projective sense) $A_0A_1A_2$ bears just the same relation to the figure as do the triangles $B_0B_1B_2$, $C_0C_1C_2$, $D_0D_1D_2$, $E_0E_1E_2$. Hence, each of the 15 perspectives of order two, for which any angular point of any one of these triangles is the fixed point and the opposite side of the same triangle the fixed line, transforms the figure into itself.

Moreover, the perspectives not only change the figure of 6 lines into itself, but they also interchange the 5 triangles among themselves. Thus, if the triangle $A_0A_1A_2$ is denoted by the letter A, each of the three perspectives considered

above leaves A unchanged, while

the first permutes B with E, and C with D,
the second permutes B with C, and D with E,
the third permutes B with D, and C with E.

The fifteen perspectives of order two then give rise to all the possible even permutations in pairs of the 5 triangles; and by carrying out these even permutations in pairs in succession, all the 60 even permutatons of the 5 triangles arise. Hence, by carrying out the fifteen perspectives of order two in suitable succession, there arise projective transformations which, while leaving the figure of 6 lines unaltered, give all possible even permutations of the 5 triangles.

Now it may be shown that a projective transformation which leaves the figure of six lines unchanged, and also leaves each triangle unchanged, necessarily leaves every point of the plane unchanged. Suppose, if possible, that such a transformation changed the line at infinity into A_1B_1. Then it necessarily changes each of the points

$$A_0,\quad B_0,\quad C_0,\quad D_0,\quad E_0,$$

into $$A_1,\quad B_1,\quad C_2,\quad D_0,\quad E_2, \qquad \text{respectively.}$$

Also, since it leaves D_0 unchanged, it must change A_1B_1 into the line at infinity; and, therefore, it permutes

$$A_0 \text{ with } A_1,\quad B_0 \text{ with } B_1,\quad C_0 \text{ with } C_2,\quad E_0 \text{ with } E_2.$$

Further, since the collineation changes A_0 into A_1, and the line at infinity into A_1B_1, it must change the other line through A_0, viz. C_1D_1 into the other line through A_1, viz. A_1E_1. This involves that it changes

$$A_0,\quad B_2,\quad C_1,\quad D_1,\quad E_2,$$

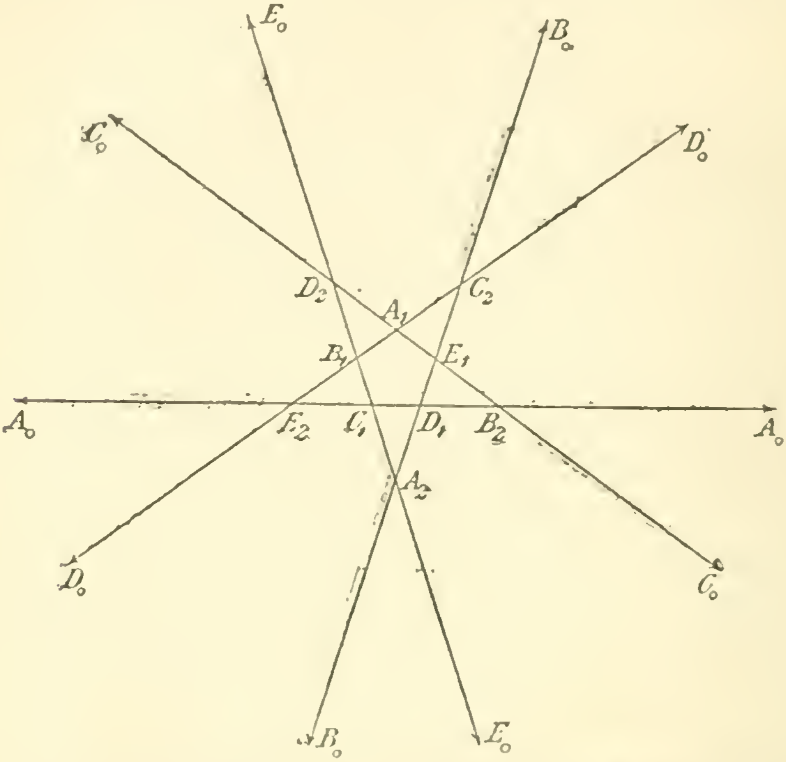
into $$A_1,\quad B_2,\quad C_0,\quad D_2,\quad E_1, \qquad \text{respectively,}$$

and it therefore permutes

$$A_0 \text{ with } A_1,\quad C_0 \text{ with } C_1,\quad D_1 \text{ with } D_2,\quad E_1 \text{ with } E_2.$$

The transformation cannot permute C_0 with C_1 and also C_0 with C_2. Hence it must leave each of the 6 lines and therefore every point unchanged. From the fifteen perspectives of order two there thus arise just 60 projective transformations (including the one which leaves each point unchanged) for which the 6 lines are permuted among themselves. Further there is a one-to-one correspondence between the 60 projective transformations and the 60 even permutations of 5 symbols.

The object of this note is to present the icosahedral group, when represented as a group of real plane collineations, in as intuitive a manner as possible; and the invariant figure



cannot, I think, be chosen in a form in which the eye more readily perceives its chief properties. From the figure given all possible forms of the figure arise by real projections. Among them is one in which the 6 lines form two equilateral triangles with a common orthocentre, corresponding vertices being on opposite sides of the orthocentre. This figure too is a very simple one, and has the advantage of not including the line at infinity in the configuration. The existence of the fifteen perspectives of order two, however, is certainly not so immediately obvious with this figure as with the one here chosen.

In conclusion, it may be noticed that, while the 15 sides of the triangles are the fixed lines of the collineations of order two, the 6 lines of the configuration are the fixed lines of the collineations of order five; and the 10 lines parallel to sides of the pentagon, of which B_1E_1 and C_2D_2 are two, are the fixed lines of the collineations of order three.

END OF VOL. XXXV.

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